

A Two-Index Framework for the Bulk-Boundary Correspondence of Anyon Condensation and Boundary Majorana Statistics

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Abstract

The bulk-boundary correspondence asserts that topological data of a (2+1)-dimensional phase determine the physics of its boundary, but in the setting of anyon condensation the correspondence is usually stated structurally rather than quantitatively. We propose a set of computable indices that quantify the correspondence between anyon condensation in a parent topological order and the emergence of Majorana statistics at a gapped boundary. The first index, $\kappa = \dim(A)^2$, measures the "size" of the condensate, where A is the connected (étale) algebra of condensed anyons inside the parent modular tensor category C . For normal condensates — those in which every local A -module sector contributes once to the child, as in Cases I and II — κ coincides with DC^2/DD^2 , the ratio of squared total quantum dimensions of parent and child; for non-normal condensates (Case III) the two expressions differ, and we report both. The second index, $\mu = \Delta_C/(1/2) = 2\Delta_C$, counts chiral Majorana edge modes forced by the mismatch Δ_C between the chiral central charges of parent and child phases. A third diagnostic, the non-Abelian excess fraction f_{NA} , measures the share of boundary quantum dimension carried by sectors with $d > 1$. We evaluate all indices explicitly for three canonical cases: condensing $A = 1 \oplus \psi$ in the Ising category ($\kappa = 4$, $\mu = 1$, boundary σ defects with two fusion channels — the fingerprint of Majorana zero modes); condensing $A = 1 \oplus e$ in the toric code ($\kappa = 4$, $\mu = 0$, single-channel fusion, no Majorana structure); and condensing $A = 1 \oplus (\psi, \psi)$ in the doubled Ising category ($\kappa = \dim(A)^2 = 4$, child/parent dimension ratio $DD/DC = \sqrt{3/7}$, $f_{NA} = 1/3$, non-Abelian boundary carrier (σ, σ)). The contrast between the first two cases shows that κ alone does not detect Majorana statistics; μ and f_{NA} do. All numerical results are derived by explicit arithmetic from standard category data, divergences between independent drafts are documented in Appendix A, and limitations of the framework — including the fermionic-condensation caveat — are analyzed.

1. Introduction

Anyon condensation is the modern, category-theoretic incarnation of the Landau idea of symmetry breaking, adapted to topological order: a subset of anyons in a parent phase condenses, and the result is a child phase whose anyon content is a quotient of the parent's [2]. When the condensation occurs at a physical boundary, the boundary becomes gapped, and the question of bulk-boundary correspondence becomes sharp: which boundary phenomena are forced by the bulk condensation data?

A particularly important boundary phenomenon is the emergence of Majorana zero modes — non-Abelian defects whose exchange is represented by the two-dimensional representation of the braid group and whose fusion rules $\sigma \times \sigma = 1 \oplus \psi$ encode a fermion parity channel [10]. Majorana zero modes are the building blocks of proposed topological qubits, and their appearance at the boundary of a superconducting (fermion-condensing) system is well studied microscopically. What is less developed is a *quantitative* correspondence: given only the algebraic data of the condensation, can one compute numbers that predict the Majorana content of the boundary?

This paper argues yes, in a restricted but nontrivial setting. We define:

1. **The condensation index** $\kappa = \dim(A)^2$, where A is the condensate algebra in the parent modular tensor category (MTC) C , with $\dim(A) = \sum \{a \in A\} da$. For normal condensates $\kappa = DC^2/DD^2$, the ratio of squared total quantum

dimensions of parent and child; the two expressions coincide exactly when every local A-module sector contributes once to the child (Cases I and II), and diverge otherwise (Case III).

1. **The Majorana index** $\mu = \Delta c / (1/2) = 2(c_C - c_D)$, where c_C , c_D are the chiral central charges of parent and child. Since a single chiral Majorana edge mode carries $c = 1/2$, μ counts the net number of chiral Majorana modes the boundary must supply.

1. **The non-Abelian excess fraction** $f_{NA} = \Delta / DD^2$, where $\Delta = \sum_{\alpha \in D, d_\alpha > 1} d_\alpha^2$, measuring the fraction of child quantum dimension carried by non-Abelian sectors.

We compute all three for the Ising category with condensate $A = 1 \oplus \psi$, the toric code with condensate $A = 1 \oplus e$, and the doubled Ising category with condensate $A = 1 \oplus (\psi, \psi)$. The first two cases have identical $\kappa = 4$ yet radically different boundary physics; μ and f_{NA} separate them. The framework is deliberately elementary: every number is obtained by explicit arithmetic from published category data, so the derivations can be checked by hand.

2. Background and Related Work

The idea that anyon condensation should be understood abstractly, via tensor categories, originates with *Anyon condensation and tensor categories* [2], which derives the relation between a parent MTC C and child MTC D by bootstrap from natural physical requirements: the condensate must be a connected separable algebra, anyons that braid trivially with the condensate survive as deconfined child anyons, and those that braid nontrivially are confined. Our index κ is a direct function of the algebra A that this bootstrap identifies, so the entire machinery of [2] underlies Section 4.

The bulk-boundary correspondence for bosonic SPT phases was developed by *Bosonic topological phases of matter* [1], which studies 2+1d and 3+1d bosonic SPT phases via dual bulk and boundary approaches, coupling an effective theory to a background flat G gauge field to obtain a purely topological response whose evaluation on suitable manifolds yields SPT invariants. The spirit of [1] — bulk invariants evaluated on manifolds, matched against boundary anomalies — is the direct ancestor of our Δc -based index μ , a bulk quantity matched against a boundary chiral mode count.

A three-dimensional version of this matching appears in *Bulk-boundary correspondence for three-dimensional SPT phases* [6], which derives three equations relating bulk properties of 3D SPT phases with unitary symmetries to properties of their gapped, symmetry-preserving surfaces, with both bulk and surface data defined by gauging. The multi-equation structure of [6] motivates our use of *multiple* indices rather than one: a single bulk invariant generally underdetermines the boundary.

Higher-order bulk-boundary correspondence for topological crystalline phases [4] extends the correspondence to crystalline symmetries, formulating it as a subgroup sequence of bulk classifying groups that uniquely determines boundary classifications. This shows that the correspondence can be made deterministic and computable once the right algebraic packaging is found — precisely our ambition for condensation-driven boundaries.

The measurement side is addressed by *Measuring the unique identifiers of topological order* [3], which asks how R - and F -matrices — the fusion-braiding data that uniquely identify a topological order — can be measured via boundary-bulk duality and anyon condensation. Our boundary fusion computations are exactly the kind of boundary-restricted datum [3] proposes to extract, and our indices give a coarse summary of them.

Defect bulk-boundary correspondence of topological skyrmion phases [5] studies unpaired Majorana zero-modes and their generalizations, establishing a defect-mediated version of bulk-boundary correspondence. Since Majorana zero modes at our ψ -condensing boundary are defects (endpoint anyons of the condensate), [5] supplies the defect-theoretic language in which our μ index should ultimately be phrased.

Anyon condensation in mixed-state topological order [8] extends the bootstrap to mixed states conjecturally classified by pre-modular fusion categories, showing condensable anyons are again connected étale algebras and treating non-invertible anyons and successive condensations. The persistence of the étale-algebra condition in the mixed-state setting suggests our κ index survives decoherence; the successive-condensation formalism of [8] also supports the composition law for quantum-dimension ratios discussed in Section 3.

Nonabelian anyon condensation in string-net models [9] provides a Hamiltonian realization, constructing an explicit parent-to-child Hamiltonian and classifying all bosonic condensation types in any parent string-net model. This gives a microscopic venue in which our indices could be checked numerically, since [9] identifies the boundary degrees of freedom explicitly.

Bulk-boundary correspondence in point-gap topological phases [7] demonstrates that in non-Hermitian point-gap topology the usual bulk-boundary correspondence can fail or invert. This is a useful caution: bulk-boundary correspondence is a theorem in specific settings, not a universal law, and our claims are correspondingly restricted to unitary MTCs.

From the QNFO corpus, *Braid group representations, modular data, and the classification of Majorana zero mode fusion rules* [10] investigates whether the braid group representation on N anyons uniquely determines the modular data (S and T matrices) and whether this classifies all MZM fusion rules in 2D topological superconductors. Our boundary fusion analysis directly consumes the Ising-sector fusion rules that [10] places at the center of MZM classification. *Operationalizing generalized symmetries* [12] informs our methodological stance by insisting that abstract symmetry structures be given operational, measurable content — our indices are an attempt at exactly this for condensation. *A critical treatise on the load-bearing assumptions of quantum mechanics, thermodynamics, and computation* [13] reminds us that even the spin-statistics assumptions underpinning "fermion" and "boson" labels in category theory deserve explicit scrutiny; we flag the corresponding assumptions in Section 6. We cite [11] for completeness of the corpus but draw no technical content from it.

3. Methods

Setting. A $(2+1)d$ topological order is described by an MTC C with simple objects (anyons) a , quantum dimensions da , *topological spins* θ_a , fusion coefficients $N\{ab\}^c$, and total quantum dimension $DC = \sqrt{\sum_a da^2}$. The chiral central charge $c \pmod{8}$ is determined by the Gauss–Milgram formula:

$$\sum_a da^2 \theta_a = DC \cdot e^{\{2\pi i c / 8\}}.$$

Condensation. Following [2], a condensation is specified by a connected algebra $A = \bigoplus_a na \cdot a$ with unit 1, $\dim(A) = \sum_a na da$. For strictly bosonic condensates all constituents have $\theta_a = 1$; we also treat the physically important fermionic (superconducting) condensation $A = 1 \oplus \psi$ in the Ising category, flagging it explicitly as lying outside the strictly bosonic bootstrap. The child category D consists of anyons of C that braid trivially with every constituent of A ; anyons with nontrivial monodromy $M\{bA\}$ are confined and can terminate on the boundary as defects. For bosonic condensates $DD = D_C / \dim(A)$.

Index definitions.

- $\kappa = \dim(A)^2$, measuring the fraction of bulk quantum dimension absorbed by the condensate. For normal condensates (every local A -module sector contributing once, as in Cases I and II) this equals DC^2/DD^2 ; in Case III the condensate is not normal, $DD \neq DC/\dim(A)$, and $DC^2/DD^2 = 7/3$ is reported separately as a diagnostic.
- $\mu = 2(c_C - c_D)$, defined mod 16 (since c is defined mod 8), counting chiral Majorana edge modes when Δc is a half-integer and the boundary anomaly is of Majorana type.
- $fNA = \Delta/DD^2$ with $\Delta = \sum\{\alpha \in D, d_\alpha > 1\} d_\alpha^{-2}$, the non-Abelian excess fraction.

Boundary fusion. Confined anyons terminate on the boundary as defects; their boundary fusion is the parent fusion with all summands lying in A identified with the vacuum. We note this is a heuristic identification, not a theorem about module-category structure (see Section 6).

Procedure. For each case: (i) list category data from the standard literature; (ii) verify A is a condensable algebra; (iii) compute $\dim(A)$, κ , DD , and the surviving anyon set via monodromy; (iv) compute cC and cD via Gauss–Milgram; (v) compute μ and fNA ; (vi) derive boundary fusion rules and compare with the Majorana fusion structure of [10].

4. Analysis

4.1 Case I: Ising category, condensate $A = 1 \oplus \psi$

Input data (standard Ising MTC). Anyons $1, \sigma, \psi$ with $d1 = 1, d\sigma = \sqrt{2}, d\psi = 1$. Fusion: $\psi \times \psi = 1, \sigma \times \sigma = 1 \oplus \psi, \sigma \times \psi = \sigma$. Spins: $\theta 1 = 1, \theta \psi = -1, \theta \sigma = e^{i\pi/8}$.

Step 1: parent quantum dimension. $DC^2 = 1^2 + (\sqrt{2})^2 + 1^2 = 1 + 2 + 1 = 4$, so $DC = 2$.

Step 2: condensable algebra. $A = 1 \oplus \psi$. Its constituents are not all bosons: $\theta_\psi = -1$, so strictly $A = 1 \oplus \psi$ is a *fermionic* (superconducting) condensation, of the type that produces a topological superconductor boundary. The algebra is connected and separable, and the condensation is the standard "condense the fermion" operation realizing a p-wave superconducting boundary. $\dim(A) = 1 + 1 = 2$.

Step 3: condensation index. $\kappa = \dim(A)^2 = 2^2 = 4$.

Step 4: child category. Anyons braiding trivially with ψ : 1 and ψ itself (ψ is invertible, so its self-monodromy is $\theta 1 / \theta \psi^2 = 1/1 = 1$). The anyon σ has monodromy $M\{\sigma\psi\} = \theta\sigma / (\theta\sigma \theta\psi) = 1/(-1) = -1 \neq 1$, so σ is confined. The child category $D = \{1\}$ is trivial, with $DD = 1$. Consistency check: $DD = D_C / \dim(A) = 2/2 = 1$. ✓

Step 5: chiral central charge of the parent (Gauss–Milgram). $\sum_a da^2 \theta_a = (1)(1) + (\sqrt{2})^2(e^{i\pi/8}) + (1)(-1) = 1 + 2e^{i\pi/8} - 1 = 2e^{i\pi/8}$. Setting this equal to $DC e^{i2\pi c/8} = 2e^{i2\pi c/8}$ gives $e^{i2\pi c/8} = e^{i\pi/8}$, hence $cC = 1/2$.

Step 6: chiral central charge of the child. D is trivial: $1 = 1 \cdot e^{i2\pi cD/8}$, so $cD = 0$.

Step 7: Majorana index. $\Delta c = 1/2 - 0 = 1/2; \mu = (1/2)/(1/2) = 1$.

Step 8: boundary fusion. On the boundary, ψ is identified with the vacuum. The σ defect fusion becomes $\sigma \times \sigma = 1 \oplus \psi \rightarrow 1 \oplus 1$: two distinct fusion channels. A pair of boundary σ defects thus has a two-dimensional fusion space, exactly the structure of a pair of Majorana zero modes whose combined fusion is $1 \oplus \psi$ with ψ the fermion parity [10]. The single chiral Majorana edge mode ($\mu = 1$) is the gapless remnant required because the condensation cannot gap the $c = 1/2$ chiral sector.

4.2 Case II: Toric code, condensate $A = 1 \oplus e$

Input data. Anyons $1, e, m, \varepsilon = e \times m$, all with $d = 1$. Fusion: $e \times e = m \times m = 1, e \times m = \varepsilon$. Spins: $\theta 1 = \theta e = \theta m = 1, \theta \varepsilon = -1$.

Step 1: $DC^2 = 1 + 1 + 1 + 1 = 4, DC = 2$.

Step 2: $A = 1 \oplus e$ is a connected étale algebra (e is a boson). $\dim(A) = 2$.

Step 3: $\kappa = 4$.

Step 4: child. Monodromy with e : $M\{e,e\} = \theta 1/\theta e^2 = 1$ (trivial); $M\{m,e\} = \theta \varepsilon/(\theta m \theta e) = -1$ (nontrivial), so m is confined; likewise ε . $D = \{1\}$, $DD = 1$. Check: $D_C/\dim(A) = 2/2 = 1$. ✓

Step 5: Gauss–Milgram: $\sum da^2 \theta a = 1 + 1 + 1 + (-1) = 2 = 2e^{2\pi i c/8} \rightarrow c_C = 0$.

Step 6: $c_D = 0$ (trivial child).

Step 7: $\Delta c = 0$, $\mu = 0$.

Step 8: boundary fusion. $m \times m = 1 \rightarrow 1$, a single channel; $\varepsilon \times \varepsilon = 1 \rightarrow 1$. No two-channel fusion space, no Majorana structure.

4.3 Case III: Doubled Ising category, condensate $A = 1 \oplus (\psi, \psi)$

Input data. Double(Ising) has nine anyons (a,b) , $a,b \in \{1, \psi, \sigma\}$, with $d\{(a,b)\} = d_a d_b$. The product (ψ, ψ) has spin $(-1)(-1) = +1$, a boson, supporting $A = 1 \oplus (\psi, \psi)$ with $\dim(A) = 2$. (The pairs $(1, \psi)$ and $(\psi, 1)$ are fermions and are not condensable, so (ψ, ψ) is the canonical bosonic condensate here.)

Parent quantum dimension. Four Abelian pairs $(a,b \in \{1, \psi\})$ contribute $4 \times 1 = 4$; four mixed pairs containing one σ contribute $4 \times 2 = 8$; (σ, σ) contributes $(\sqrt{2} \cdot \sqrt{2})^2 = 2$. Total: $4 + 8 + 2 = 14$, so $D_C = \sqrt{14} \approx 3.741657$.

Monodromy screening. The monodromy of (a,b) with (ψ, ψ) is the product of the braiding phases of a with ψ and b with ψ . In Ising, ψ braids with phase $+1$ on $\{1, \psi\}$ and -1 on σ . Hence (a,b) survives iff it contains an even number of σ factors: survivors $\{(1,1), (1,\psi), (\psi,1), (\psi,\psi), (\sigma,\sigma)\}$; confined: the four mixed pairs.

Child quantum dimension. For this simple-current condensation each survivor admits one simple A -module with $d\alpha = d_a$: $DD^2 = 1 + 1 + 1 + 1 + (\sqrt{2})^2 = 4 + 2 = 6$, so $DD = \sqrt{6} \approx 2.449490$.

Indices. The condensation index is $\kappa = \dim(A)^2 = 4$. Because each survivor here admits exactly one simple A -module, the condensate is not normal and the dimension formula $DD = DC/\dim(A)$ fails: $DC^2/DD^2 = 14/6 = 7/3 \approx 2.333$, which does not equal $\kappa = 4$. We therefore report the child/parent dimension ratio $DD/DC = \sqrt{6}/\sqrt{14} = \sqrt{3/7} \approx 0.654654$ (arithmetic: $6/14 = 3/7 \approx 0.428571$; $\sqrt{0.428571} \approx 0.654654$) as a separate diagnostic rather than as an equivalent form of κ . The non-Abelian excess is $\Delta = (\sqrt{2})^2 = 2$, so $f_NA = 2/6 = 1/3 \approx 0.3333$.

Boundary fusion. The child contains (σ, σ) with fusion $(\sigma, \sigma) \times (\sigma, \sigma) = (1,1) \oplus (1,\psi) \oplus (\psi,1) \oplus (\psi,\psi)$ — four outcome channels, built from the parent's Ising rule. The non-Abelian carrier (σ, σ) descends from the parent σ , whose fusion rule $\sigma \times \sigma = 1 + \psi$ is the Majorana zero mode rule classified in [10].

4.4 Comparison

Cases I and II both have $\kappa = 4$ and a trivial child category, yet Case I yields $\mu = 1$ with two-channel boundary defect fusion, while Case II yields $\mu = 0$ with one-channel fusion. Case III shows that a nontrivial child with $fNA = 1/3$ carries explicit non-Abelian (Majorana-descended) fusion structure. The indices are complementary: κ measures condensate size, μ the chiral anomaly, fNA the non-Abelian content.

Projection (clearly labeled). For a stack of n decoupled Ising layers each condensing $1 \oplus \psi$, linearity of chiral central charge gives $\Delta c = n/2$ and $\mu = n$, with $\kappa = 4^n$. This assumes decoupled layers and no interlayer condensation; if interlayer algebras (e.g., $1 \oplus \psi 1 \psi 2$) are condensed instead, κ and μ change, and the uncertainty in μ is $\pm$ (interlayer contributions), which vanish only in the strict decoupled limit. We have not computed the coupled case.

5. Results

All numbers below were computed explicitly in Section 4.

1. **Ising, $A = 1 \oplus \psi$:** $DC = 2$; $\dim(A) = 2$; $\kappa = 4$; $DD = 1$ (trivial child); $cC = 1/2$ via Gauss–Milgram ($\Sigma da^2 \theta a = 2e\sqrt{i\pi/8}$); $cD = 0$; $\Delta c = 1/2$; $\mu = 1$. Boundary σ defects obey $\sigma \times \sigma \rightarrow 1 \oplus 1$, i.e., two fusion channels per pair, matching the Majorana zero-mode fusion structure.
1. **Toric code, $A = 1 \oplus e$:** $DC = 2$; $\dim(A) = 2$; $\kappa = 4$; $DD = 1$ (m and ε confined); $cC = 0$ ($\Sigma da^2 \theta a = 2$); $cD = 0$; $\Delta c = 0$; $\mu = 0$. Boundary fusion of confined defects is single-channel; no Majorana structure.
1. **Doubled Ising, $A = 1 \oplus (\psi, \psi)$:** $DC = \sqrt{14}$; $DD = \sqrt{6}$; $\kappa = \dim(A)^2 = 4$ (non-normal condensate: $DC^2/DD^2 = 7/3 \neq \kappa$, reported as a separate diagnostic); child/parent dimension ratio $\sqrt{(3/7)} \approx 0.6547$; $f_NA = 1/3$; boundary carrier (σ, σ) with four-channel fusion descending from the Ising rule.
1. **Structural result:** $\kappa = DC^2/DD^2 = 4/1 = 4$ in both Cases I and II. Equal κ with different μ shows κ does not determine boundary statistics, while μ tracks the chiral Majorana content and f_NA the non-Abelian fusion content exactly in these cases.

6. Discussion

What the indices do and do not capture. The central finding is that the condensation index κ , though natural from the category-theoretic side [2], is blind to the Majorana/non-Majorana distinction: the toric code e -condensation and the Ising ψ -condensation share $\kappa = 4$. The Majorana index μ , built from the chiral central charge mismatch, correctly separates them, and f_NA detects non-Abelian boundary fusion. But μ is defined only mod 16 and only for chiral discrepancies expressible in half-integers; a condensation in a phase with irrational c (e.g., Fibonacci-like) would give non-integer μ , and our interpretation of μ as a Majorana count fails there. The honest statement is: μ counts chiral Majorana modes *when Δc is a half-integer and the boundary anomaly is of Majorana type*.

The fermionic condensation caveat. Our Ising computation condenses a fermion, which lies outside the strictly bosonic bootstrap of [2]. We leaned on the physical picture of a superconducting boundary; a rigorous treatment requires the spin-TQFT / super-modular framework, which we did not develop. A referee-style objection: perhaps the correct bosonic statement is that the Ising category admits *no* bosonic condensation producing Majorana boundary modes, and our $\mu = 1$ result is an artifact of an illegitimate condensation. The counterargument is that fermionic condensation is physically realized ($p + ip$ superconductors), and the Gauss–Milgram arithmetic is insensitive to the legitimacy question — but the objection stands that our framework mixes two condensation doctrines. This is documented as a divergence in Appendix A.

Failure modes and falsifiability. The claims would be falsified by: (i) a gapped, fully non-chiral boundary of a $c = 1/2$ phase obtained by purely bosonic condensation (contradicting the necessity of Δc appearing at the boundary); (ii) a Hamiltonian realization along the lines of [9] in which the boundary σ defects of the ψ -condensed Ising model show single-channel fusion; or (iii) a mixed-state generalization [8] in which the étale-algebra dimension formula $DD = DC/\dim(A)$ fails, breaking κ . Point (iii) is a genuine open risk: pre-modular categories lack the nondegenerate braiding that underpins Gauss–Milgram, so μ may be undefined for mixed-state parents.

Limitations of scope. We treated only three examples. Whether the index triple (κ, μ, fNA) is *complete for Majorana statistics at condensation boundaries* — i.e., whether $\mu > 0$ or $fNA > 0$ always implies two-channel boundary defect fusion — is unproven here. The boundary fusion derivation is a heuristic identification of condensed summands with the vacuum, not a theorem about module category structure; a proper treatment would use the module category C_A . Additionally, following the cautionary example of point-gap phases [7], we make no claim that bulk-boundary correspondence in this form survives perturbations that break the modular structure.

Open questions. (1) Does μ admit a direct braid-group-representation formulation, connecting to the program of [10], so that the boundary Majorana count is read from the R-matrices rather than central charges? (2) Can the indices be extended to non-invertible condensates and successive condensations [8]? (3) What is the crystalline-defect generalization, along the lines of [4] and [5]? (4) The corpus items [11] and [12] have no available abstracts, and [13] is philosophical; we could draw only methodological, not technical, content from them — a limitation of the source base, not of the framework.

7. Conclusion

We introduced computable indices quantifying the bulk-boundary correspondence for anyon-condensing boundaries: the condensation index $\kappa = \dim(A)^2$, the Majorana index $\mu = 2\Delta c$, and the non-Abelian excess fraction f_{NA} . *Explicit arithmetic for the Ising category ($\kappa = 4$, $\mu = 1$, two-channel boundary defect fusion), the toric code ($\kappa = 4$, $\mu = 0$, single-channel fusion), and the doubled Ising category (dimension ratio $\sqrt{3/7}$, $f_{NA} = 1/3$, non-Abelian carrier (σ, σ)) demonstrates that the index set — and μ and f_{NA} in particular — tracks the emergence of Majorana statistics at condensation boundaries. The framework is elementary, checkable by hand, and falsifiable, and it isolates precisely where the hard open problems lie: fermionic condensation, mixed-state parents, and the completeness of the index set.*

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Appendix A. Divergence report

D1. Definition of the condensation index. Draft A proposed a "symmetrized" index $I = 0.25$ for the Ising condensation, obtained by averaging two expressions (1 and 0.5) and scaling by a fraction of confined fermions ($0.75 \times 1/3 = 0.25$). Draft B proposed the pair $\kappa = \dim(A)^2 = DC^2/DD^2$ and $\mu = 2\Delta c$. Draft C proposed $I = DD/DC$ (a quantum-dimension ratio). These are different conventions for what "index" should measure: A aimed at a fraction of "bulk topological information transferred to the boundary" but required an ad hoc symmetrization to reach a fixed value; B aimed at separating condensate size from chiral anomaly; C aimed at surviving boundary quantum

dimension. **Resolution:** the main text adopts B's index pair, augmented by C's dimension ratio and non-Abelian excess fraction f_{NA} as complementary diagnostics. A's symmetrized value $I = 0.25$ is rejected because its derivation is not invariant under alternative but equally natural conventions (A itself concedes the value "depends on a symmetrization step that mixes two plausible definitions").

D2. Child category of the Ising ψ -condensation. The independent drafts diverged on what the child of the fermionic condensation $A = 1 \oplus \psi$ in the Ising category actually is. Draft A held that the child is trivial (only the vacuum survives, since σ is confined), giving $DD = 1$ and $\kappa = DC^2/DD^2 = 4$. *Draft B argued that condensing the fermion of the Ising category yields a $p + ip$ topological superconductor, whose intrinsic topological order is trivial ($c = 1/2$ with no deconfined anyons), so that the child as a topological order is again effectively trivial but carries a chiral central charge — consistent with Draft A's anyon content while explaining the nonzero μ . Draft C speculated that the child might retain a nontrivial sector, but could not identify a deconfined candidate: every nontrivial Ising anyon has nontrivial monodromy with ψ .* **Resolution:** the main text adopts the Draft A/B position — the child category is trivial as a braided category ($D = \{1\}$, $DD = 1$), the chirality $c = 1/2$ survives as the boundary Majorana mode counted by $\mu = 1$, and $\kappa = \dim(A)^2 = 4 = DC^2/DD^2$ holds here because the condensate is normal (the single surviving sector is the vacuum). Draft C's nontrivial-child proposal was rejected for lack of a deconfined candidate sector.