

Discriminating the Arithmetic Cut: Matched-Level-Density Null Models for the Primon-Gas Specific Heat and Spectral Correlations

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Abstract. The arithmetic-statistics program reads the two exchange statistics of identical particles as two multiplicity rules on one integer lattice, and its published observable is the specific heat of the primon gas: a computable deviation from the smooth ideal gas at every sampled temperature. The program's own red team raised the decisive objection: any irregular spectrum with the same level density might reproduce that deviation, making the observable level-density content rather than arithmetic content. This paper adjudicates the objection computationally. It constructs three matched-level-density non-arithmetic null ensembles, realizes each under the same statistics as the cut, and computes the specific heat and the unfolded two-point statistics of the cut against them. The specific heat separates from the fixed-count nulls at every tested cutoff (z-scores from 5 to 100) and from the Poisson-type nulls beyond a computable minimum cutoff; a focused test of the small-spacing exclusion separates at 34 sigma. The full two-point curve does not separate under a uniform distance measure at any tested cutoff, which locates the arithmetic information in the small-spacing hard core. The paper also adjudicates two published numbers in the program: the reported Dyson number-variance pair mixes two window lengths and uses an asymptotic formula outside its validity range, and the Bost-Connes specific heat at inverse temperature 1.06 is a finite-cutoff crossover whose approach to the pole is power-law in the cutoff. The premises end where a physical temperature is identified at a p-adic place; nothing in this paper crosses that boundary.

1. The objection this paper answers

A line of work reads the two exchange statistics of identical particles as the two multiplicity rules of one integer lattice. Unrestricted exponents give the Riemann zeta function and Bose-Einstein occupation; the squarefree restriction gives a ratio of zeta values and Fermi-Dirac occupation [quni2026stats, quni2026adelic], and a bounded-occupation family interpolates between them without carrying an exchange phase [quni2026anyons]. The published observable of that identification is thermodynamic: the specific heat of the primon gas deviates from a smooth-density ideal gas at every sampled temperature, by up to roughly three quarters at low temperature [quni2026anyons].

That observable was never tested against the null it must defeat. An experimentalist who engineers a spectrum of prime-logarithmic mode energies and measures a specific-heat deviation has no way, from the published record alone, to tell the deviation of an arithmetic spectrum from the deviation of any irregular spectrum with the same level density. This paper builds that test. It constructs the matched-density non-arithmetic null ensembles, runs the arithmetic cut against them under the same statistics, and reports the separation thresholds in the cutoff and the observable.

2. The arithmetic cut and its thermodynamics

Fix a cutoff P and let the modes be the primes $p \leq P$ with energies $\varepsilon_p = \ln p$ and inverse temperature β . Three statistics are realized by three occupation rules, each with an exact closed form for the specific heat:

Bose (unrestricted): $SCV(B) = \beta^2 \sum (\ln p)^2 p^{-\beta} (1-p^{-\beta})^{-2}$.

Fermi (squarefree): $SCV(F) = \beta^2 \sum (\ln p)^2 p^{-\beta} (1+p^{-\beta})^{-2}$.

Maxwell-Boltzmann: $C_V^{(MB)} = \beta^2 \sum_p (\ln p)^2 p^{-\beta}$, whose logarithmic partition function is the prime zeta function $\zeta_P(\beta) = \sum_p p^{-\beta}$. The primon gas and its arithmetic-gas variants are thirty-year-old objects of the statistical theory of numbers [julia1990statistical, bakas1991curiosities], and the bounded-occupation family studied here extends them in the program's register [quni2026adelic].

Two limits are exact. At high temperature every Bose mode contributes one unit of specific heat (equipartition over $\pi(P)$ modes) while the Fermi and Boltzmann modes decay to zero. At low temperature all three statistics collapse to $\beta^2 (\ln 2)^2 2^{-\beta}$, the freeze-out of the lowest mode. All of this is verified in the deposited scripts, which reproduce the equipartition limit to 10^{-6} and the collapse exactly.

The ideal-gas baseline used by the published claim is $\pi(P)$ units of specific heat, flat in temperature. The relative deviation $D(\beta) = 1 - C_V^{(B)}(\beta)/\pi(P)$ is monotone: it reaches three quarters at $\beta \approx 0.42$ (about 2.4 ideal-gas temperatures) and approaches 100 percent at low temperature. The phrase "up to roughly three quarters at low temperature" is therefore window-dependent: the deviation exceeds three quarters for all lower temperatures and tends to full freeze-out, so the figure in the published abstract is correct only for a specific, previously unstated temperature window.

3. The matched-density null ensembles

Three null families share the arithmetic cut's level count $N = \pi(P)$ and range $[\ln 2, \ln P]$ but carry no prime structure:

- **Smooth log-spaced.** Levels evenly spaced in log-energy. Deterministic; the fluctuation-free comparator.
- **Fixed-count random.** N independent uniform points on the log range, sorted. Matched count and range, irregular, no arithmetic structure.
- **Poisson-on-log-scale.** A Poisson number of points with the same mean density. Matched mean density with count fluctuations.

Fairness is enforced at two levels. First, every null is realized under the same statistics as the cut: the null levels play the role of the modes in the closed forms of Section 2. Second, the two-point observables use each family's own smooth staircase (the exact log-prime count for the cut, the linear-in-log count for the nulls), the same binning, the same window, and matched sample counts. Without these two rules, a spacing artifact is mistaken for arithmetic content; with them, the comparison isolates the arithmetic structure itself.

4. Validation of the machinery

Before any discrimination result, the estimators must recover known answers. The deposited suite validates:

- **The GUE pair correlation** from Monte Carlo samples of the Hermitian Gaussian ensemble, matched to the analytic bulk curve to within 0.07 mean absolute deviation, and the estimator passes exact sanity checks (a uniform grid gives number variance exactly zero; an exact Poisson sample gives 10.63 ± 0.1 against the expected 10 at window 10).
- **Montgomery-Odlyzko.** The first 3000 Riemann zeros, unfolded by the Riemann-von Mangoldt smooth count, match the GUE pair correlation to 0.11 mean absolute deviation with the expected repulsion at small spacing [montgomery1973pair, odlyzko1987distribution].

- **The twin-gap hard core.** The minimum unfolded prime spacing is exactly

$2/\ln P$ (the minimum prime gap is 2; beyond the hard core the primes are Poisson-like [gallagher1976distribution]): measured 0.206128 against the predicted 0.206099 at $P = 2^{14}$, with zero spacings below the threshold.

- **A positive control.** A planted log-periodic modulation of the level

density is recovered by the spectral form factor at the planted frequency to 8×10^{-5} relative error.

5. Discrimination results

For each cutoff $P \in \{2^8, 2^{12}, 2^{16}\}$, each statistics, and each null family, the cut's specific-heat curve and two-point curve are compared to the null family's own self-distance distribution (30 realizations), and separation is reported in units of its standard deviation.

Specific heat. The cut separates at or above two sigma in every statistics against the fixed-count nulls from the smallest tested cutoff onward (z-scores from 4.9 at $P = 2^8$ to 102 at $P = 2^{16}$). Against the Poisson-type nulls the separation is not significant at the smallest cutoff (0.28 for Bose), becomes significant at $P = 2^{12}$ (4.1), and grows to 63 at $P = 2^{16}$. The minimum cutoff for a two-sigma specific-heat verdict is therefore $P \approx 2^8$ against fixed-count nulls and $P \approx 2^{12}$ against Poisson-type nulls, for all three statistics. The disconfirmation criterion this machine serves is pre-registered for the program's estimation record [quni2026hd3, quni2026ump014].

Two-point statistics. Under a uniform distance measure over the unfolded pair-correlation curve, the cut does not separate from the nulls at any tested cutoff (z-scores near minus one). This is expected physics: beyond the hard core the primes are asymptotically Poisson [gallagher1976distribution], so the full curve carries little beyond the level density. The arithmetic information is concentrated in the small-spacing exclusion: the fraction of nearest-neighbor spacings below $2/\ln P$ is exactly zero for the cut and 0.1656 ± 0.0049 for the fixed-count nulls, a separation of 34 sigma at $P = 2^{16}$. Any two-point verdict must report both numbers; the uniform measure alone would understate the channel.

The three null families overlap at the smallest cutoff (their shared mean density forces it) and separate from each other as the cutoff grows — a consistency check, not a claim.

6. Two published numbers adjudicated

The Dyson number-variance pair. The program's estimation paper reports "a Dyson number variance 1.044 against the predicted 0.525" [quni2026ump014]. Both numbers are values of the Dyson asymptotic formula $(1/\pi^2)[\ln(2\pi L) + 1 + \gamma - \pi^2/8]$ at different window lengths: 1.0449 at $L = 3400$ and 0.5246 at $L = 20$. The sentence compares two different windows. Independently of that mismatch, the asymptotic formula itself is not accurate at these lengths: the exact two-point reduction $\Sigma^2(L) = L - 2 \int_0^L (L-s)(\sin \pi s/\pi s)^2 ds$ [dyson1962statistical, mehta2004random] gives 0.65 at $L = 20$ against the asymptotic 0.52, a 24 percent underestimate, and the GUE Monte Carlo samples match the exact reduction within 8-13 percent. At the low height of the first 3000 zeros the measured values exceed the exact reduction by 1.2-2.8 times, growing with the window; that residual is the known height-dependent unfolding error of the Riemann-von Mangoldt count [odlyzko1987distribution].

The Bost-Connes specific heat. The estimation paper reports a Bost-Connes critical specific heat of 316.3 at $\beta = 1.06$ against a predicted pole amplitude 312.1 [quni2026ump014]. The pole amplitude is recovered exactly by the deposited computation, but the truncated-product specific heat at $\beta = 1.06$ is 33.1 at $P = 10^4$, 47.5 at 10^5 , and 62.7 at 10^6 : the tail $\sum_{p>P} p^{-\beta}$ decays only as $P^{-(\beta-1)}$, which is about 0.44 even at $P = 10^6$, so the

published 316.3 cannot come from a direct truncated-product computation at any feasible cutoff [bost1995hecke]. The honest observable is the finite-cutoff crossover table reported here; the pole language is reserved for the infinite limit. Ensemble averaging over Hamiltonians, as in the randomized Riemann gas, is known to wash out the pole entirely [duenas2014thermodynamics], so no such averaging is applied to the thermodynamic observable.

7. What a practitioner can do

A spectroscopist or metrologist considering an engineered prime-logarithmic spectrum gets a discrimination machine rather than a conclusion. The deposited scripts accept any level sequence and a cutoff, generate the three matched-density null families under the chosen statistics, and return: the separation z-scores per observable, the minimum cutoff at which a two-sigma specific-heat verdict is available, and the small-spacing exclusion test. The same machine converts the published specific-heat deviation into a threshold table usable as an engineering specification: a qudit register or optical lattice with $P \approx 2^{12}$ or more modes is sufficient to discriminate the arithmetic cut from matched non-arithmetic spectra in the thermodynamic channel, and the hard-core test decides the two-point channel. The protocol is blind to the origin of the spectrum, so it applies to any candidate arithmetic system.

8. Limitations

The cutoffs tested run to $P = 2^{16}$ (6542 modes); larger cutoffs and noise models (temperature stability, resolution) are extensions, not claims. The molecular data applications discussed elsewhere in the program require desymmetrization by species and per-species unfolding before any of these tests apply [quni2026ump014]; the machine here is validated on the cut and on controlled ensembles. The low-height number-variance residual of Section 6 is quantified empirically and attributed to the known unfolding error; its exact decomposition is left open. The finite-p-base primon gas has an independent kernel-theoretic treatment [franchini2024padic], and the conformal primon gas of the Belinski-Khalatnikov-Lifshitz setting generalizes the partition functions studied here [hartnoll2025conformal]; neither changes the null construction of this paper. The statistical-class claims are about mathematical structure: no physical realization is asserted, and the premises end where a physical temperature is identified at a p-adic place.

9. Reproducibility

Every quantitative statement in this paper is reproduced by a deposited, deterministic script (seed 20260829, Python 3.12.10, NumPy 2.4.4, SciPy 1.17.1), with the outputs in the record's verification directory: the estimator suite (pair correlation, number variance against the exact two-point reduction, form factor, GUE samples, planted control) and the discrimination suite (thermodynamics, null ensembles, separation tables, hard-core test). The Riemann zeros are computed by a vectorized Riemann-Siegel method. The exact-theory integral is evaluated numerically at each reported window length.

References
