

The Corrected Primon-Gas Dictionary: Zeta Partition Functions and the Discipline of Arithmetic Interpretations

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2026-08-29

Abstract. The primon gas — a free quantum gas whose single-particle modes are indexed by the primes, with energies equal to the logarithms of the primes — has connected quantum statistical mechanics with multiplicative number theory since the early 1990s and remains in active use in high-energy theory. This paper consolidates the correspondence into a single audited reference: a term-by-term dictionary in which every entry is corrected and every formula is verified by deposited deterministic computations, a five-level ladder that separates mathematical isomorphism from physical realization claims, and a negative list stating what the correspondence does not license. The dictionary is exact and model-specific. The premises end where a physical temperature would be identified at a p-adic place, and no such identification is asserted.

Keywords. [primon gas, zeta function, partition function, quantum statistics,

1. Introduction

The primon gas is a free quantum gas whose single-particle modes are labelled by the primes: mode p carries energy $\varepsilon_p = \ln p$, the many-body states are labelled by the integers $n = \prod_p p^{a_p}$, and the Hamiltonian is multiplication by $\ln n$. Its grand canonical partition function is the Riemann zeta function, and its statistics — unrestricted, squarefree, or bounded occupation — generate a small family of exact identities between thermodynamic quantities and multiplicative number theory. The construction goes back to the statistical reading of the zeta function [julia1990], the supersymmetric reformulation of the Möbius inversion [spector1990], and the study of arithmetic gases [bakas1991]; its deepest version is the Bost–Connes system, where the symmetry-breaking phase transition at inverse temperature one carries the arithmetic of the maximal abelian extension of the rationals [bostconnes1995]. The correspondence is not dormant: it is currently used in cosmology, where modular-invariant states near a spacelike singularity define dual primon gases [hartnoll2025] and complex primon gases built from the Gaussian and Eisenstein integers [declerck2025], and in the statistical mechanics of mean-field spin glasses, where the gas acquires a kernel representation [franchini2024].

This paper does not claim the correspondence as new. Its contribution is a consolidation with discipline: a corrected, audited dictionary in which every entry is stated exactly and every formula is verified by deposited deterministic computations; a five-level ladder that separates what the correspondence is (an exact mathematical isomorphism) from what it is not (a physical realization claim); a ledger of corrections to errors that have circulated in informal drafts of the dictionary; and a negative list stating what the correspondence does not imply. Earlier work in the same research program established the squarefree origin of the Fermi–Dirac/Bose–Einstein distinction [adelicquantumstatistics2026], the bounded-occupation family with its absent exchange phase [arithmeticanyons2026], the consolidated map with its practitioner crosswalk [adelicquantumarithmetic2026], the computational discrimination of the arithmetic cut from matched-density nulls [arithmeticcutdiscrimination2026], and the realization-independent hierarchy distance that underlies the whole construction [distinctionbasedultrametric2026].

2. The Correspondence

Setup. Single-particle modes are labelled by primes p with energies $\varepsilon_p = \ln p$; many-body states are labelled by integers $n = \prod_p p^{a_p}$ with occupation exponents a_p ; the Hamiltonian acts by $(\hat{H}f)(n) = (\ln n) f(n)$. The inverse temperature β is a formal parameter; its identification with the complex variable s of the zeta function is a choice on the real section, flagged throughout as formal.

Partition functions. The grand canonical partition functions are exact:

$$Z_B(\beta) = \prod_p (1 - p^{-\beta})^{-1} = \zeta(\beta),$$

for unrestricted occupation ($a_p \in \mathbb{N}_0$);

$$Z_F(\beta) = \prod_p (1 + p^{-\beta}) = \frac{\zeta(\beta)}{\zeta(2\beta)},$$

for squarefree occupation ($a_p \in \{0, 1\}$); and

$$\ln Z_{MB}(\beta) = \sum_p p^{-\beta} = P(\beta),$$

for the distinguishable gas, where P is the prime zeta function. The bounded-occupation (Gentile) family interpolates:

$$Z_m(\beta) = \prod_p \frac{1 - p^{-(m+1)\beta}}{1 - p^{-\beta}},$$

with $m = 1$ reproducing the Fermi gas and $m \rightarrow \infty$ the Bose gas. No exchange phase appears anywhere in the family: an occupation cap is not a braid phase, and the phases that standard anyon models carry are multiplicative characters at roots of unity — a different arithmetic object [arithmeticanyons2026]. The three free-gas statistics are transforms of the prime zeta function:

$$\ln Z_B = \sum_{k \geq 1} \frac{P(k\beta)}{k}, \quad \ln Z_F = \sum_{k \geq 1} \frac{(-1)^{k+1} P(k\beta)}{k}, \quad \ln Z_{MB} = P(\beta).$$

A chemical potential, i.e. a fugacity $z = e^{\beta\mu}$, deforms the Bose product to

$$Z_\mu(\beta) = \prod_p (1 - z p^{-\beta})^{-1} = \sum_n z^{\Omega(n)} n^{-\beta},$$

where $\Omega(n)$ counts prime factors with multiplicity. This is the z -weighted generating function of the integers, *not* a Dirichlet L -function: a Dirichlet character twists each Euler factor as $\prod_p (1 - \chi(p)p^{-s})^{-1} = L(s, \chi)$, a multiplicative phase filter, which is a different object from a chemical potential.

Thermodynamic observables. With $U = -\partial_\beta \ln Z$,

$$U_B = \sum_p \frac{\ln p}{p^\beta - 1}, \quad U_F = \sum_p \frac{\ln p}{p^\beta + 1},$$

and the specific heat carries the full derivative factor:

$$C_V = -\beta^2 \partial_\beta U, \quad C_V^B = \beta^2 \sum_p \frac{(\ln p)^2 p^{-\beta}}{(1 - p^{-\beta})^2}, \quad C_V^F = \beta^2 \sum_p \frac{(\ln p)^2 p^{-\beta}}{(1 + p^{-\beta})^2}.$$

The entropy is $S = \ln Z + \beta U$, which for the Bose gas reads

$$S = \sum_p \left[-\ln(1 - x_p) + \beta (\ln p) \frac{x_p}{1 - x_p} \right], \quad x_p = p^{-\beta}.$$

Zeros as fluctuations, not definitions. The level-count function $\psi(x) = \sum_{n \leq x} \Lambda(n)$ obeys the explicit formula

$$\psi_0(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \ln(2\pi) - \frac{1}{2} \ln(1 - x^{-2}),$$

so the nontrivial zeros of the zeta function enter as subleading oscillatory corrections to the smooth count. They do not define the statistics and they do not define the leading thermodynamics. The zeros themselves follow the GUE two-point law $R_2(s) = 1 - (\sin \pi s / \pi s)^2$ [montgomery1973, odlyzko1987]; the primes are Poisson-like beyond a hard core [gallagher1976]: consecutive primes differ by at least two (for primes at least three), which is a minimum unfolded spacing of $2/\ln p$ — a first bin of the spacing histogram that is exactly empty below that width. The sharp small-spacing exclusions that discriminate the prime spectrum from random sets therefore test the primes, not the zeros.

Operators and phase structure. The Hamiltonian is the multiplication operator by $\ln n$, with $\text{Tr} e^{-\beta \hat{H}} = \zeta(\beta)$; the von Mangoldt function is a *coefficient*, entering through $-\zeta'(s)/\zeta(s) = \sum_n \Lambda(n) n^{-s}$, not an operator of the model. The power ζ^k corresponds to k independent copies of the gas, not to k -body interactions. The pole of $\zeta(\beta)$ at $\beta = 1$ is the infinite-mode limit of the free gas, realized in the Bost–Connes system as a genuine symmetry-breaking transition [bostconnes1995]. At any finite prime cutoff, there is no singularity, only a smooth crossover; a numerical evaluation point, such as $\beta = 1.06$, is a probe near the would-be pole, not a phase-transition temperature of any finite system.

3. The Correction Ledger

The following corrections repair errors that circulated in an informal draft dictionary of the correspondence. Each row states the erroneous form and the corrected form; every corrected formula is verified in code (Section 6).

(a) *Modes and states.* The draft wrote $\varepsilon_i \equiv \ln n_i$, conflating levels with states. The single-particle modes are the primes with energies $\ln p$; the many-body states are the integers $n = \prod_p p^{a_p}$ with energies $\ln n$. The many-body level count up to energy E is $\lfloor e^E \rfloor$ (integers), whereas the single-particle mode count is the prime-counting function; the two are different objects.

(b) *Chemical potential versus character.* The draft mapped $e^{\beta\mu}$ to a Dirichlet character. The fugacity gives the z -weighted generating function of Section 2; a character is a twist of the Euler factors. They are not the same object.

(c) *The Maxwell–Boltzmann row.* The draft wrote a Boltzmann row with a labelling factor inconsistent with the unification rule. The consistent statement is $\ln Z_{MB} = P(\beta)$.

(d) *Specific heat.* The draft defined $C_V = \partial_\beta U$. The correct definition is $C_V = -\beta^2 \partial_\beta U$; the missing factor and sign are restored in the formulas above.

(e) *Entropy.* The draft's entropy had a wrong sign on the first term and a dimensionally wrong second term. The correct form carries $\beta \ln p$ in the second term, as above.

(f) *The small-spacing exclusion.* The draft attributed the sharp small-spacing exclusion of the prime spectrum to GUE repulsion of the zeros. The exclusion tests the primes (twin-gap hard core); the zeros' GUE behaviour is a separate statement.

(g) *Phase transitions and continuation.* The draft read the pole of the zeta function as a physical transition of a finite system and analytic continuation as the thermodynamic limit. The pole is the infinite-mode limit; continuation extends the function and corresponds to no finite system's partition function.

(h) *Eigenstates.* The draft called the primes "energy eigenstates." The eigenstates of $\hat{H}$ are the integers.

(i) *The Hamiltonian.* The draft identified the Hamiltonian with an arithmetic derivative. It is multiplication by $\ln n$; the von Mangoldt function enters as a coefficient of $-\zeta'/\zeta$.

(j) *Interactions.* The draft read Dirichlet convolution as an interaction and ζ^k as k -body interactions. Convolution is a generating-function parallel, not a Hamiltonian term, and ζ^k is k independent species.

(k) *Observables.* The draft's "Theorem 2" bundled the specific-heat signature with the small-spacing exclusion. They are different observables with different nulls: the specific heat is thermodynamic, the exclusion is a two-point statistic of the prime spectrum [arithmeticcutdiscrimination2026].

4. The Five-Level Interpretive Ladder

Claims of the form "arithmetic structure appears in physics" are not one claim. They stratify into five levels.

- **L0 — distinction.** The marking of an inside against an outside. A

methodological primitive: the framework treats it as unanalysable, which is a choice of starting point, not an ontic commitment; the re-entrant calculus developed this primitive into a formal system [reentrantdistinctions2026].

- **L1 — hierarchy.** The distinction-based ultrametric: the number of

distinctions required to separate two states. Definitional and realization-independent; arithmetic enters only when a hierarchy is dressed in prime or p -adic clothing [@distinctionbasedultrametric2026; @ultrametricprogram2026]. The counting construction that turns distinctions into quantities — quantity as broken idempotence — is prior work [idempotentcore2026], and is credited rather than re-derived here.

- **L2 — isomorphism.** Euler products, zeta identities, and the

bounded-occupation family. An exact mathematical isomorphism; the content of Section 2 lives here.

- **L3 — statistical hypothesis.** Physical spectra carry arithmetic

correlations beyond universal random-matrix statistics. A falsifiable distributional claim, requiring a pre-registered null.

- **L4 — physical instantiation.** A specified system realizes the

arithmetic partition function. A claim about a laboratory system, held to the same protocol discipline that the finite-distinction reading of quantum mechanics applies to its own claims [finitedistinctionqm2026].

The ladder carries three inference rules. First, *L2 cannot imply L4*: the fact that a formal partition function equals the zeta function does not place any physical system at L4. Second, **L3 is the only admissible bridge**: a physical claim must be stated as a distributional prediction with a pre-registered null. Third, *L4 requires a protocol*: a specified spectrum, a specified counting rule, a pre-registered null, and a pre-registered test.

Two honest statements about these rules. The rules are methodological discipline, not a discovery: the L2-to-L4 inference they forbid is an inference that informal drafts of this very correspondence made, so the rules are non-vacuous as self-correction, and as external guidance they are a pre-commitment — "admissible inference" is defined by the protocol above, and the ladder's falsifiability is precisely that an exhibited admissible L2-to-L4 inference would break its central rule. And the ladder is a classificatory device, not a theory: it cannot be falsified by data, only outperformed or abandoned. This paper's empirical content is inherited, not new: the discrimination results on which the L3 statements rest are published elsewhere, including a confirmed separation of the arithmetic cut from matched-density nulls and a disconfirmed specific-heat-only separation, adjudicated as pre-registered [arithmeticcutdiscrimination2026].

5. What the Correspondence Licenses, and What It Does Not

The negative list. The exactness of the dictionary does not license: a derivation of spin-statistics from the Euler product; a universe made of primes; an identification of Riemann zeros with measured energy levels; or evidence for the Hilbert–Pólya programme. The zeros enter through the explicit formula as fluctuations; they define neither the statistics nor the leading thermodynamics. Where the informal draft preamble spoke of "the physical universe and the mathematical universe as two dialects of the same statistical language," the precise statement is narrower: a free quantum gas on a prime-logarithmic spectrum is combinatorially and analytically isomorphic to multiplicative number theory — the primes provide the modes, the integers provide the many-body states, the zeta function provides the partition function — and that isomorphism is exact in the toy model and silent beyond it.

What a practitioner can do. Two concrete artifacts follow from the dictionary. First, a specification for an engineered log-prime spectrum: a device (superconducting registers, optical lattices, or photonic arrays) whose mode frequencies are proportional to $\ln p$, whose occupation caps implement the Gentile family of Section 2, and whose readout follows the corrected thermodynamic formulas; every formula needed for the readout is verified rather than asserted (Section 6). Second, a scope statement that decides, *before* an experiment, what a realization claim may consist of: by the ladder's rules, a claim that a device realizes the arithmetic partition function must state the spectrum, the counting rule, the null model, and the test in advance — an engineering-relevant discipline that separates a physical signature from a simulation.

6. Verification

Every quantitative statement in Sections 2 and 3 is verified by two deposited, deterministic computation suites (52 checks in total, all passing, released with this paper).

The first suite verifies the dictionary identities at $\beta = 2$: the three partition functions against $\zeta(2) = \pi^2/6$, $\zeta(2)/\zeta(4)$ and the prime zeta value $P(2) = 0.45224742\dots$, with explicit truncation-tail corrections (the tail at prime cutoff 10^6 is computed with the three-term expansion of the exponential integral, not the one-term form, which errs by several percent); the unification expansions to $k = 30$; the Gentile limits $m = 1 \rightarrow$ Fermi and $m \rightarrow \infty \rightarrow$ Bose; the specific-heat definition against a finite-difference computation of $-\beta^2 \partial_\beta U$; the entropy against $\ln Z + \beta U$; and the fugacity identity $Z_\mu = \sum_n z^{\Omega(n)} n^{-\beta}$ against a direct sum over integers.

The second suite verifies the spectral and analytic statements: the explicit formula $\psi_0(x)$ at $x = 20$ and $x = 30$ against the exact summatory von Mangoldt function, using 120 exact zero ordinates (residuals 0.018 and 0.020); a seeded Monte Carlo of the Gaussian unitary ensemble (120 matrices of size 150, semicircle unfolding — no rank unfolding), whose two-point correlation matches $1 - (\sin \pi s / \pi s)^2$ in the bulk with maximum deviation 0.036 and whose number variance at window lengths 5, 10 and 20 matches the exact two-point reduction $\Sigma^2(L) = L - 2 \int_0^L (L - s)(\sin \pi s / \pi s)^2 ds$ to within the stated tolerance (the Monte Carlo windows are

centred on a grid of arbitrary positions; counting around data points instead would measure the Palm count, whose mean is $2 \int_0^{L/2} R_2(t) dt$, not L — a subtlety the deposited code documents and avoids); the number variance at $L = 20$ and $L = 3400$ against the Dyson asymptotic $(1/\pi^2) [\ln(2\pi L) + 1 + \gamma - \pi^2/8]$, which converges from below with a relative deficit of 20–33% over $L \leq 50$; the exact logarithmic-integral unfolding $\text{Li}(x) = \text{Ei}(\ln x)$ against the known values $\text{Li}(2) = 1.0451637801\dots$ and $\text{Li}(10^6) = 78627.54916\dots$, together with a demonstration that the asymptotic series for Li is unusable at small argument; the Fermi observables against finite differences; the identity $\text{Tr} e^{-\beta \hat{H}} = \zeta(2)$ with tail correction; the von Mangoldt convolution $\Lambda = \log * \mu$; ζ^k as k independent species; and the twin-gap hard core as a *computed* bin count — among 78,496 unfolded spacings of primes below 10^6 , the first bin is empty while a continuous Poisson null expects several thousand, a hard-core deficit of $z = -86.4$.

The anchors of the published lineage are recovered with attribution: $\beta^2/(\beta - 1)^2 = 312.111$ at $\beta = 1.06$ is the analytic pole-amplitude value; the exact recomputed specific heat at $\beta = 1.06$ is ≈ 311.9 (finite sum to prime cutoff 10^7 plus analytic tail); and the value **316.3** that circulated earlier is *not* the exact value — it is a finite-difference artifact of a coarse computation, and is treated as an adjudication target only.

Two data notes follow from the re-computation. First, a zero-ordinate cache deposited with an earlier study of this program [arithmeticcutdiscrimination2026] was found, on re-computation against independent exact values, to be a coarse approximation with maximum error ≈ 0.38 ; the suites here use exact values instead, and downstream users of that cache should re-derive the zeros rather than reuse it. Second, the prime-spacing distribution at mid-range shows a large deviation from a continuous Poisson reference ($z = +27.5$ in one bin): prime gaps are even and alternate modulo six, so a continuous Poisson process is the wrong reference there; the correct nulls are the matched-level-density ensembles of the discrimination study, which are out of scope here and declared as such.

Reproducibility statement. The suites are deterministic; the seeded Monte Carlo uses seeds 20260829 and 777. Runtime: Python 3.12.10, NumPy 2.4.4, SciPy 1.17.1, mpmath (exact zero ordinates), on Windows x64; wall-clock for the full second suite is minutes on a laptop. The scripts and their outputs are deposited with this paper; every number in this section is produced by running them.

7. Premise Boundaries

Where the premises end, stated plainly. The identification $\beta = s$ is a formal choice, flagged throughout; nothing here identifies a physical temperature at any p -adic place. The completeness of the correction ledger is an audit-level statement, not a proof: a twelfth error found by an independent reader would be a corrigendum, not a collapse, and that status is asserted rather than concealed. The empirical content is inherited from the discrimination study and is not re-claimed here. And the central honesty of the ladder, stated once: the correspondence is verified-exact at L2, and nothing in that exactness moves the L3/L4 needle.

8. Term Crosswalk

For the reader arriving from either side, the correspondence in one table.

Quantum statistics	Multiplicative number theory
Single-particle mode p , energy $\ln p$	Prime p
Many-body state, energy $\ln n$	Integer $n = \prod_p p^{a_p}$
Occupation exponent a_p	Prime exponent in the factorization
Bose gas (unrestricted occupation)	All integers; $\zeta(\beta)$

Quantum statistics	Multiplicative number theory
Fermi gas (squarefree occupation)	Squarefree integers; $\zeta(\beta)/\zeta(2\beta)$
Boltzmann gas	Prime zeta function $P(\beta)$
Gentile family (occupation cap m)	Exponents bounded by m
Fugacity $z = e^{\beta\mu}$	Weight $z^{\Omega(n)}$; not a character
Specific heat $C_V = -\beta^2 \partial_\beta U$	Prime-weighted variance of $\ln p$
Level-count oscillations	Explicit formula; zeta zeros as corrections
Small-spacing exclusion	Twin-gap hard core of the primes

Changelog

- v2.0.0: Adversarial audit revision. Fixes: prose, prose.