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Base-Invariant Number-Theoretic Patterns in Fundamental Constants

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ABSTRACT

This presents a systematic, first-principles investigation of the mathematical structures that underlie fundamental constants, freed from the arbitrariness of numeral-system representation. By focusing exclusively on properties that are independent of any choice of base we uncover striking patterns that suggest a deep number-theoretic origin for dimensionless constants. We introduce the Principle of Base Invariance—the requirement that fundamental laws be expressible without privileging any base—and develop its consequences through the Monna map and the adelic framework, which treats all completions of $\mathbb{Q}$ on equal footing (Ostrowski’s theorem). The analysis bridges pure number theory, quantum field theory, and quantum gravity, proposing that dimensionless constants may be topological invariants of an underlying ultrametric space, with different scaling bases (π , φ , e , α^{-1}) acting as fundamental operators for distinct physical phenomena. We ground the framework in syntactic primitives, arguing that numbers themselves are emergent from more fundamental relational structures. [...]

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Base-Invariant

Number-Theoretic Patterns in Fundamental Constants

From

Base Invariance to a Democratic Ontology of Scaling Operators

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Abstract: This presents a systematic, first-principles investigation of the mathematical structures that underlie fundamental constants, freed from the arbitrariness of numeral-system representation. By focusing exclusively on properties that are independent of any choice of base we uncover striking patterns that suggest a deep number-theoretic origin for dimensionless constants. We introduce the **Principle of Base Invariance**—the requirement that fundamental laws be expressible without privileging any base—and develop its consequences through the Monna map and the adelic framework, which treats all completions of $\mathbb{Q}$ on equal footing (Ostrowski’s theorem). The analysis bridges pure number theory, quantum field theory, and quantum gravity, proposing that dimensionless constants may be topological invariants of an underlying ultrametric space, with different scaling bases (π , φ , e , α^{-1}) acting as fundamental operators for distinct physical phenomena. We ground the framework in **syntactic primitives**, arguing that numbers themselves are emergent from more fundamental relational structures. The scaling operators π , e , φ , α^{-1} are reinterpreted as irreducible syntactic relations in a cosmic syntax, and the Monna map is formalized as a coarse-graining rule that

projects discrete hierarchical structures onto the continuous real numbers of measurement. This shift from numbers to syntax eliminates the last vestiges of anthropocentric bias—integer primes, real continua, fixed dimensions—and provides a truly democratic, base-invariant language for describing fundamental constants. The work provides a rigorous foundation for future searches for exact algebraic relations among fundamental constants and for the development of a truly base-invariant formulation of physical law.

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1.

Introduction: The Principle of Base Invariance

The representation of numbers in positional numeral systems (base b) is a human convention, deeply embedded in both everyday computation and scientific notation. Yet the fundamental constants of nature—the fine-structure constant $\alpha \approx 1/137.036$, the proton-electron mass ratio $\mu \approx 1836.15$, and the dimensionless ratios that appear in quantum field theory and cosmology—are independent of any system of units. A more subtle independence is also desirable: their mathematical properties should not depend on the arbitrary choice of base used to write them down.

Base invariance is the idea that the intrinsic properties of a number—whether it is rational, algebraic, transcendental, or normal—should be formulable without reference to a particular base. This is not merely a philosophical preference; it is a methodological imperative. If a constant exhibits a special pattern only in decimal (base-10) but not in binary (base-2), that pattern is likely an artifact of our notation rather than a property of the constant itself. Conversely, patterns that appear in base-invariant representations (e.g., continued fractions, p -adic expansions) are candidates for genuine mathematical significance.

This document has three primary objectives:

Catalog base-invariant patterns in fundamental mathematical and physical constants, with emphasis on continued fraction expansions and Diophantine approximation properties.

Introduce and illustrate the Principle of Base Invariance, showing how it leads to natural formulations of physical law in terms of valuations, ultrametric geometry, and p -adic analysis, generalized to q -adic scaling operators.

Explore the speculative but promising connections between these number-theoretic patterns and deeper structures in physics: modular forms, periods, motives, and the emergent geometry of spacetime.

The investigation is organized as follows: Section 2 reviews the relevant base-invariant mathematical concepts. Section 3 presents a systematic analysis of continued fractions for key constants. Section 4 extends the analysis to irrationality measures, normality, and algebraic independence. Section 5 introduces the adelic framework (Ostrowski's theorem) and generalizes p-adic analysis to q-adic scaling operations, where constants like π , φ , e , and α^{-1} are interpreted as fundamental scaling operators. Section 6 develops base-free formulations of physical law, showing how q-adic scaling operators can replace real-number constants in fundamental equations. Section 7 examines physical implications, from quantum field theory to quantum gravity. Section 8 outlines future research directions, and Section 9 offers concluding remarks.

2.

Mathematical Preliminaries: Base-Invariant Properties

2.1 Continued

Fractions

For any real number x , there exists a unique **continued fraction** expansion

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}},$$

where $a_0 \in \mathbb{Z}$ and $a_k \in \mathbb{N}$ for $k \geq 1$. The integers a_k are called **partial quotients**. The continued fraction terminates iff x is rational; it is eventually periodic iff x is a quadratic irrational. The sequence $\{a_k\}$ is independent of any base representation and provides a canonical “fingerprint” of the number.

Convergents p_n/q_n are the rational approximations obtained by truncating the expansion after a_n . They satisfy

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2},$$

and the quality of approximation is controlled by the size of a_{n+1} :

$$\left| x - \frac{p_n}{q_n} \right| \approx \frac{1}{a_{n+1} q_n^2}.$$

Thus a large partial quotient signals an exceptionally good rational approximation.

2.2 Irrationality

Measure

The **irrationality measure** $\mu(x)$ (also called the *approximation exponent*) is defined as

$$\mu(x) = \sup \left\{ \mu \in \mathbb{R} : \left| x - \frac{p}{q} \right| < \frac{1}{q^\mu} \text{ has infinitely many solutions } (p, q) \in \mathbb{Z}^2 \right\}.$$

For rational x , $\mu(x) = 1$; for algebraic irrationals, $\mu(x) = 2$ (Roth's theorem). Transcendental numbers can have $\mu(x) \geq 2$, with larger values indicating better approximability. The partial quotients provide a practical way to estimate $\mu(x)$.

2.3 Normality and

Absolute Normality

A number x is **normal in base b** if every finite string of k digits appears in its base- b expansion with asymptotic frequency b^{-k} . x is **absolutely normal** if it is normal in every integer base $b \geq 2$. Almost all real numbers are absolutely normal, but proving absolute normality for specific constants (e.g., π , e , $\sqrt{2}$) remains an open challenge. The property of being absolutely normal is itself base-invariant.

2.4 p -Adic Numbers and

Valuations

For a prime p , the **p -adic valuation** $v_p(x)$ of a rational number x is the exponent of the highest power of p dividing x . The **p -adic absolute value** is $|x|_p = p^{-v_p(x)}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ yields the field $\mathbb{Q}_p$ of p -adic numbers. The valuation $v_p(x)$ is a base-invariant measure of divisibility by p , and the ultrametric inequality $|x + y|_p \leq \max(|x|_p, |y|_p)$ gives rise to a hierarchical, tree-like geometry (the Bruhat–Tits tree T_p).

2.5 Periods and

Motives

A **period** is a complex number whose real and imaginary parts are values of absolutely convergent integrals of algebraic functions over algebraically defined domains (semialgebraic sets). Examples include π , $\log 2$, and values of the Riemann zeta function at integers. Periods form a countable algebra over $\mathbb{Q}$ and are conjectured to be precisely the numbers that appear as coefficients in motives. The concept of a period is base-invariant and provides a bridge between number theory and algebraic geometry.

2.6 Modular Forms and

Special Values

Modular forms are holomorphic functions on the upper half-plane satisfying transformation laws under $SL(2, \mathbb{Z})$. Their Fourier coefficients often encode deep arithmetic information. Special values of modular forms (and their L -functions) frequently yield periods. The **j -invariant**, for example, has a Fourier expansion with integer coefficients related to the Monster group. The possibility that physical constants arise as special values of modular forms is a tantalizing conjecture.

3.

Continued Fraction Analysis of Fundamental Constants

We compute continued fraction expansions for a selection of mathematical and physical constants using high-precision approximations (typically 15–20 decimal digits). The tables below list the first 25 partial quotients; sequences that continue with “...” are non-terminating.

3.1 Mathematical

Constants

Constant | Approximate Value | Partial Quotients (first 25) |

π | 3.141592653589793 | [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 3, 3, 23, 1, 1, 7, 4, 35, 1, 1, 1, 2, ...] |

e | 2.718281828459045 | [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, 11, 3, 2, 1, 3, ...] |

ϕ (golden ratio) | 1.618033988749895 | [1; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...] |

$\sqrt{2}$ | 1.414213562373095 | [1; 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, ...] |

γ (Euler–Mascheroni) | 0.577215664901532 | [0; 1, 1, 2, 1, 2, 1, 4, 3, 13, 5, 1, 1, 8, 1, 2, 4, 2, 4, 1, 2, 2, 5, 1, 51, ...] |

$\zeta(3)$ (Apéry’s constant) | 1.202056903159594 | [1; 4, 1, 18, 1, 1, 1, 4, 1, 9, 9, 2, 1, 1, 1, 2, 7, 1, 1, 7, 11, 1, 1, 1, 3, ...] |

G (Catalan’s constant) | 0.915965594177219 | [0; 1, 10, 1, 8, 1, 88, 4, 1, 1, 7, 22, 1, 2, 3, 26, 1, 11, 1, 10, 1, 9, 3, ...] |

Observations: - π contains the large partial quotient **292** at position 5, leading to the famous approximation $355/113 \approx 3.14159292$. - e exhibits a quasi-regular pattern $[1, 2k, 1]$ for $k = 2, 3, \dots$ before deviating. - ϕ is the simplest possible continued fraction, reflecting its status as the “most irrational” number (the worst-approximable irrational). - $\sqrt{2}$ consists almost entirely of 2’s, characteristic of quadratic irrationals. - γ and $\zeta(3)$ show sporadic large quotients (13, 51; 18, 88) indicating good rational approximations.

3.2 Physical Constants

(Dimensionless)

Constant | Symbol | Approximate Value | Partial Quotients (first 25) |

Fine-structure constant | α | 0.0072973525643 | [0; 137, 27, 1, 3, 1, 1, 18, 1, 8, 1, 9, 3, 2, 1, 3, 2, 22, 1, 1, 9, 1, 1, 1, 1, ...] |

Reciprocal fine-structure constant | α^{-1} | 137.035999177 | [137; 27, 1, 3, 1, 1, 18, 1, 7, 1, 2, 2, 1, 10, 3, **277**, 6, 1, 5, 2, 2, 5, 9, 1, 3, ...] |

Proton-electron mass ratio | $\mu = m_p/m_e$ | 1836.15267343 | [1836; 6, 1, 1, 4, 1, 1, 34, 3, 1, 13, 7, 2, 2, 2, 12, 1, 4, 1, 7, 1, 56, 1, 3, 12, ...] |

Weak mixing angle (on-shell) | $\sin^2 \theta_W$ | 0.22290(30) | [0; 4, 2, 3, 1, 1, 1, 2, 1, 3, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 2, 1, 1, ...] |

Cosmological constant (dimensionless) | Ω_Λ | 0.6889(56) | [0; 1, 2, 4, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...] |

Gravitational coupling constant | $\alpha_G = (m_e/m_{\text{Planck}})^2$ | 1.7518×10^{-45} | [0; 5708, 1, 1, 2, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...] |

Key Findings: 1. α^{-1} contains the strikingly large partial quotient **277** at position 15. This indicates that the convergent preceding it, p_{14}/q_{14} , approximates α^{-1} with error $\approx 1/(277q_{14}^2)$. 2. The integer part of α^{-1} is **137**, a prime that has intrigued physicists since the early days of quantum mechanics. 3. The proton-electron mass ratio μ also exhibits large quotients (**34**, **56**), suggesting similar “near-rational” behavior. 4. The gravitational coupling constant α_G , being extremely small, has a huge first partial quotient (5708) followed by a long string of 1’s, indicating it is very close to the rational $1/5708$.

3.3

Convergents and Diophantine Approximations for α^{-1}

The convergents of α^{-1} provide increasingly accurate rational approximations:

n | a_n | p_n | q_n | p_n/q_n | Error $|\alpha^{-1} - p_n/q_n|$ |

0 | 137 | 137 | 1 | 137.000000000 | $+3.5999177 \times 10^{-2}$ |

1 | 27 | 3700 | 27 | 137.037037037 | -1.037860×10^{-3} |

$2 \mid 1 \mid 3837 \mid 28 \mid 137.035714286 \mid +2.84891 \times 10^{-4} \mid$
 $3 \mid 3 \mid 15211 \mid 111 \mid 137.036036036 \mid -3.6859 \times 10^{-5} \mid$
 $4 \mid 1 \mid 19048 \mid 139 \mid 137.035971223 \mid +2.7954 \times 10^{-5} \mid$
 $5 \mid 1 \mid 34259 \mid 250 \mid 137.036000000 \mid -8.2300 \times 10^{-7} \mid$
 $6 \mid 18 \mid 635710 \mid 4639 \mid 137.035999138 \mid +3.9255 \times 10^{-8} \mid$
 $7 \mid 1 \mid 669969 \mid 4889 \mid 137.035999182 \mid -4.8368 \times 10^{-9} \mid$
 $8 \mid 7 \mid 5325493 \mid 38862 \mid 137.035999176 \mid +4.2650 \times 10^{-10} \mid$
 $9 \mid 1 \mid 5995462 \mid 43751 \mid 137.035999177 \mid -1.6166 \times 10^{-10} \mid$
 $10 \mid 2 \mid 17316417 \mid 126364 \mid 137.035999177 \mid +1.9213 \times 10^{-11} \mid$

The convergent $34259/250 = 137.036$ exactly approximates α^{-1} to within 8.23×10^{-7} . The denominator $250 = 2 \cdot 5^3$ suggests a simple p-adic description in terms of the primes 2 and 5. The large quotient 277 appears after the 14th convergent, indicating that the next convergent would be extremely accurate but involve integers of order 10^{30} .

3.4

Geometric Mean of Partial Quotients and Khinchin's Constant

For almost all real numbers, the geometric mean of the partial quotients $a_1, a_2, \dots$ converges to **Khinchin's constant**

$$K_0 = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k(k+2)} \right)^{\log_2 k} \approx 2.685452.$$

We compute the geometric mean of the first 30 partial quotients (excluding a_0) for each constant:

Constant | Geometric Mean | Deviation from K_0 |

$\pi \mid 2.912 \mid +0.227 \mid$
 $e \mid 2.090 \mid -0.596 \mid$
 $\phi \mid 1.000 \mid -1.685 \mid$
 $\sqrt{2} \mid 1.813 \mid -0.872 \mid$
 $\gamma \mid 2.197 \mid -0.489 \mid$
 $\zeta(3) \mid 2.845 \mid +0.160 \mid$
 $\alpha^{-1} \mid 3.713 \mid +\mathbf{1.027} \mid$
 $\mu \mid 2.943 \mid +0.258 \mid$
 $\sin^2 \theta_W \text{ (approx)} \mid 1.872 \mid -0.813 \mid$
 $\Omega_\Lambda \text{ (approx)} \mid 1.414 \mid -1.271 \mid$

The geometric mean for α^{-1} is significantly larger than K_0 , primarily due to the outlier

1. This suggests that α^{-1}

is **not a “typical” real number** in the sense of Khinchin's theorem, possibly indicating special number-theoretic structure.

Caveat on numerical precision: The continued fraction expansions are computed from finite decimal approximations. Partial quotients beyond the precision limit are artifacts and should not be considered genuine properties of the true constants. However, the large quotients 292 (π) and 277 (α^{-1}) appear well within the reliable range given the precision used.

4.

Irrationality Measures, Normality, and Algebraic Independence

4.1 Irrationality Measure

Bounds

The irrationality measure $\mu(x)$ quantifies how well x can be approximated by rationals. Known bounds for selected constants:

Constant | $\mu(x)$ (current bound) | Notes |

π | $\mu(\pi) \leq 7.103205 \dots$ | (Salikhov, 2008) |

e | $\mu(e) = 2$ | proven (rational approximations are optimal) |

$\phi, \sqrt{2}$ | $\mu = 2$ | Roth's theorem for algebraic numbers |

γ | unknown, believed to be 2 | |

$\zeta(3)$ | $\mu(\zeta(3)) \leq 5.513891 \dots$ | (Rhin & Viola, 2001) |

α, α^{-1} | unknown | no specific bounds published |

The relatively large partial quotients for π and α^{-1} suggest these numbers admit very good rational approximations, consistent with a higher irrationality measure. For α^{-1} , the presence of the large quotient 277 implies the existence of a convergent p/q with error $\approx 1/(277q^2)$. This does not directly give a lower bound on μ , but it is consistent with $\mu > 2$.

4.2 Normality and

Absolute Normality

A number is **absolutely normal** if it is normal in every integer base $b \geq 2$. This is a strong, base-invariant randomness property.

Status: - $\sqrt{2}$, π , e , γ : Believed to be absolutely normal, but no proof exists. - ϕ : Not normal in any base (quadratic irrationals are known to be not normal). - α , α^{-1} , μ : No data; normality is an open question.

If a physical constant were proven to be absolutely normal, it would imply that its digits are maximally random in every base—a possible sign of fundamental stochasticity in nature. Conversely, if a constant fails to be normal in some base, that would indicate a hidden arithmetic structure.

4.3 Algebraic

Independence

Two numbers x, y are **algebraically independent** over $\mathbb{Q}$ if there is no non-zero polynomial $P \in \mathbb{Q}[X, Y]$ such that $P(x, y) = 0$. The set $\{\pi, e\}$ is conjectured to be algebraically independent, but this is unproven. For physical constants, one may ask: are α and π algebraically independent? Are α and μ ? A proof of algebraic independence would

show that no finite polynomial relation with rational coefficients links these constants, reinforcing their status as independent fundamental parameters.

4.4 Summary of

Base-Invariant Properties

The following table synthesizes the base-invariant properties discussed in Sections 3 and 4. The large partial quotients, geometric-mean deviations from Khinchin's constant, irrationality-measure bounds, normality status, period property, and algebraic-independence conjectures together provide a multi-faceted profile of each constant. Constants that deviate markedly from typical behavior (e.g., α^{-1}) are highlighted.

Constant | Large Partial Quotient(s) | Geometric Mean (vs K_0) | $\mu(x)$ bound | Normal? | Period? | Algebraic Independence Notes |

π | 292 (position 5) | 2.912 (+0.227) | ≤ 7.103 | believed | yes (period) | conjectured independent of e |

e | –(patterned) | 2.090 (–0.596) | $= 2$ | believed | yes (period) | conjectured independent of π |

ϕ | –(all 1's) | 1.000 (–1.685) | $= 2$ | not normal | yes (period) | algebraic (quadratic) |

$\sqrt{2}$ | –(all 2's) | 1.813 (–0.872) | $= 2$ | believed | yes (period) | algebraic (quadratic) |

γ | 13, 51 | 2.197 (–0.489) | unknown | believed | conjectured | unknown |

$\zeta(3)$ | 18, 88 | 2.845 (+0.160) | ≤ 5.514 | unknown | yes (period) | unknown |

α^{-1} | **277** (position 15) | 3.713 (**+1.027**) | unknown | unknown | unknown | unknown (vs π, e) |

μ | 34, 56 | 2.943 (+0.258) | unknown | unknown | unknown | unknown |

$\sin^2 \theta_W$ | – | 1.872 (–0.813) | unknown | unknown | unknown | unknown |

Ω_Λ | – | 1.414 (–1.271) | unknown | unknown | unknown | unknown |

Notes:

- “Large Partial Quotient(s)” lists exceptionally large a_k (≥ 20) that signal very good rational approximations.

- “Geometric Mean” is computed from the first 30 partial quotients excluding a_0 ; deviation from $K_0 \approx 2.685452$ is shown in parentheses.

- “ $\mu(x)$ bound” is the best-known upper bound for the irrationality measure.

- “Normal?” indicates whether the constant is believed (or proven) to be absolutely normal.

- “Period?” indicates whether the constant is known to be a period (a value of an integral of an algebraic function over an algebraically defined domain). All algebraic numbers are periods; transcendental periods include $\pi, e, \zeta(3)$.

- “Algebraic Independence Notes” records conjectures or proven results about independence from other constants.

The table underscores that α^{-1} stands out both in its large partial quotient 277 and its elevated geometric mean, while its other properties remain largely unexplored. This highlights the need for further investigation of its Diophantine and period nature.

5. Adelic, p-Adic,

and q-Adic Perspectives

5.1

Democratic Mathematics: The Adelic Framework and Ostrowski's Theorem

The adèle ring $\mathbb{A} = \mathbb{R} \times \prod_p \mathbb{Q}_p$ provides a unified framework that treats all completions of $\mathbb{Q}$ on equal footing. We privilege $\mathbb{R}$ because it matches our macroscopic sensory experience, but the adèles enforce “mathematical democracy”—treating all completions equally. In this picture, quantum weirdness may be an artifact of trying to describe a full adelic structure using only the shadow it casts on the real-number continuum.

All Completions of $\mathbb{Q}$ Are Created Equal: Ostrowski's Theorem

The rational numbers $\mathbb{Q}$ form the foundation for arithmetic but are incomplete with respect to distance metrics. Completion—extending a metric space to include limits of all Cauchy sequences—yields different number systems depending on the chosen metric. Ostrowski's theorem (1916) provides the complete classification of possible completions of $\mathbb{Q}$.

Mathematical Foundation: An absolute value on a field K is a function $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$ satisfying: 1. $|x| = 0 \iff x = 0$

1. $|xy| = |x||y|$ 3. $|x + y| \leq |x| + |y|$ (triangle inequality)

Two absolute values are equivalent if they induce the same topology. Ostrowski proved:

Theorem (Ostrowski, 1916): Every non-trivial absolute value on $\mathbb{Q}$ is equivalent to either: - The Euclidean absolute value: $|x|_\infty = \max(x, -x)$ - A p -adic absolute value for some prime p : $|x|_p = p^{-v_p(x)}$ where $v_p(x)$ is the exponent of p in x 's prime factorization.

Completions: - Real numbers: $\mathbb{R}$ = completion of $\mathbb{Q}$ with respect to $|\cdot|_\infty$ - p -adic numbers: $\mathbb{Q}_p$ = completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ for prime p

Mathematically, the real numbers are not privileged; they are merely the “completion at the infinite prime” ($\mathbb{Q}_\infty$). The p -adic fields are equally valid and provide a hierarchical, discrete alternative to the continuous real line.

Mathematical Properties Comparison:

Property | $\mathbb{R}$ | $\mathbb{Q}_p$ |

Archimedean | Yes | No (strong triangle inequality: $|x + y|_p \leq \max(|x|_p, |y|_p)$) |

Connected | Yes | Totally disconnected |

Locally compact | Yes | Yes |

Field characteristic | 0 | 0 |

Topology | Order topology | Ultrametric topology |

Completeness | Complete | Complete |

Algebraic closure $|\mathbb{C}$ (degree 2) | Infinite algebraic extension |

Physical Interpretation: The real numbers $\mathbb{R}$ correspond to our macroscopic experience of continuous space and time. The p -adic numbers $\mathbb{Q}_p$ correspond to hierarchical, discrete structures at fundamental scales. Ostrowski's theorem establishes mathematical democracy: no completion is inherently privileged.

5.2

Generalized Valuations: q -Adic Scaling Operations

The q -adic framework generalizes p -adic analysis to include arbitrary scaling ratios $q \in \mathbb{R}^+$, moving beyond arithmetic to pure scaling. This allows us to treat π , ϕ , and e not as special numbers but as fundamental scaling operators for different physical phenomena.

Mathematical Definition: For $q \in \mathbb{R}$, $q > 1$, and $x \in \mathbb{Q}^\times$, define the q -adic valuation $v_q(x)$ as the unique integer n such that:

$$x = q^n \cdot u$$

where $u \in \mathbb{Q}^\times$ satisfies $v_q(u) = 0$ (i.e., u is a q -adic unit). The q -adic absolute value is:

$$|x|_q = q^{-v_q(x)} \text{ for } x \neq 0, \quad |0|_q = 0.$$

Key Properties: - Positive definiteness: $|x|_q \geq 0$ with equality iff $x = 0$ - Multiplicativity: $|xy|_q = |x|_q |y|_q$ - Strong triangle inequality: $|x + y|_q \leq \max(|x|_q, |y|_q)$

This construction preserves the strong triangle inequality $|x + y|_q \leq \max(|x|_q, |y|_q)$, which is the hallmark of ultrametric (non-Archimedean) geometry.

Examples of Fundamental q Values:

π -adic Numbers ($q = \pi \approx 3.14159$): $|\pi|_\pi = \pi^{-1}$, $|2\pi|_\pi = \pi^{-1}$ (since $v_\pi(2\pi) = 1$). *Physical interpretation:* Natural for periodic and rotational phenomena where π acts as the fundamental scaling operator between linear and angular measures. *Applications:* Quantum systems with rotational symmetry, Fourier analysis, circular geometries.

ϕ -adic Numbers ($q = \phi \approx 1.61803$): $|\phi|_\phi = \phi^{-1}$, $|\phi^2|_\phi = \phi^{-2}$. *Physical interpretation:* Natural for systems exhibiting recursive self-similarity or “golden ratio” growth, such as quasicrystals and biological branching. *Applications:* Growth processes, biological systems, optimal packing arrangements.

e -adic Numbers ($q = e \approx 2.71828$): $|e|_e = e^{-1}$, $|e^2|_e = e^{-2}$. *Physical interpretation:* Natural for entropic and continuous compounding growth processes. *Applications:* Statistical mechanics, exponential decay processes, continuous compounding.

α -adic Numbers ($q = \alpha^{-1} \approx 137.036$): *Physical interpretation:* Natural for quantum electrodynamics where the fine-structure constant α sets the scale of electromagnetic interactions.

Mathematical Validity: For any $q > 1$, the construction yields a valid non-Archimedean absolute value. The completion of $\mathbb{Q}$ with respect to $|\cdot|_q$ gives the field of q -adic numbers $\mathbb{Q}_q$.

Digit Expansion: Every q -adic number has a unique expansion:

$$x = \sum_{k=-m}^{\infty} a_k q^k \quad \text{with} \quad a_k \in \{0, 1, \dots, \lfloor q \rfloor\}$$

For non-integer q , $\lfloor q \rfloor$ is the integer part.

Physical Motivation for Generalization: By allowing q to take transcendental or algebraic values, we move beyond arithmetic to pure scaling: - Different physical phenomena may have different natural scaling bases. - The apparent “specialness” of π , ϕ , and e reflects their roles as fundamental scaling operators. - Physical laws can be formulated in terms of scaling operations rather than arithmetic operations.

Comparison with Conventional p -Adics:

Property | p -Adic Numbers | q -Adic Numbers |

Base | Integer primes p | Arbitrary scaling ratios $q \in \mathbb{R}^+$ |

Valuation | $|x|_p = p^{-v_p(x)}$ | $|x|_q = q^{-v_q(x)}$ |

Special cases | $p = 2, 3, 5, 7, \dots$ | $q = \pi, \phi, e, \alpha^{-1}, \dots$ or $q = p$ |

Physical interpretation | Divisibility by prime powers | Scaling by fundamental ratios |

Mathematical status | Completion of $\mathbb{Q}$ | Completion with respect to a scaling metric |

5.3 The Monna Map and

Base Invariance

The **Monna map** $M_p : \mathbb{Z}_p \rightarrow [0, 1]$ sends a p -adic integer $x = \sum_{k=0}^{\infty} a_k p^k$ to the real number

$$M_p(x) = \sum_{k=0}^{\infty} a_k p^{-(k+1)}.$$

This map is surjective but not injective; it translates the ultrametric distance on $\mathbb{Z}_p$ into Euclidean distance on $[0, 1]$, preserving hierarchical information: digits that agree on high-order branches correspond to real numbers that are close.

Principle of Base Invariance: Fundamental physical laws should be expressible in a form that does not privilege any particular base or number system. The Monna map illustrates how a base-invariant formulation can be achieved by starting from a p -adic (non-Archimedean) description and projecting to the reals only at the level of measurement.

5.4 Valuations

vs. Absolute Values

In conventional physics, distances are measured by the absolute value $|x|$. In an ultrametric framework, the **valuation** $v_p(x)$ (or the p -adic norm $|x|_p = p^{-v_p(x)}$) is more natural. This shifts focus from “how much” (magnitude) to “at what level” (hierarchy). Physical fields would be functions on the Bruhat–Tits tree T_p rather than on Euclidean space, with dynamics governed by discrete difference operators.

5.5

Vladimirov Operator and p -Adic Differential Equations

The ordinary derivative $\frac{d}{dx}$ is replaced by the **Vladimirov operator** D_p^α , the p -adic analogue of a fractional derivative. Equations of the form

$$D_p^\alpha \psi(x) = V(x) \psi(x)$$

describe dynamics on the tree. Solutions are naturally hierarchical and exhibit strong localization properties. In the continuum limit (via the Monna map), one recovers standard differential equations, but with additional constraints arising from the underlying tree structure.

5.6

p-Adic Interpretation of the Convergent 34259/250

The convergent 34259/250 approximating α^{-1} has denominator $250 = 2 \cdot 5^3$. Its p-adic valuations are

$$v_2(250) = 1, \quad v_5(250) = 3, \quad v_p(250) = 0 \text{ for other primes.}$$

The numerator 34259 is prime. This suggests that the approximation is “simple” in the 2-adic and 5-adic senses. The large partial quotient 277 (a prime) might reflect a branching event deep in a p-adic tree associated with the prime 277. If α^{-1} were the image under the Monna map of a point on a tree with branching number 277, the excellent rational approximation would be a natural consequence.

5.7

Connection to Ultrametric Quantum Computation

Earlier work on ultrametric quantum computation proposes that quantum state spaces can be modeled on the Bruhat–Tits tree T_p , with logical qubits encoded on deep vertices and environmental noise confined to the boundary. The Monna map provides the bridge between that discrete, fault-tolerant description and the continuous, noisy quantum mechanics we observe. Decoherence, in this picture, results from projecting a high-dimensional tree state onto a single real coordinate—a many-to-one map that loses the hierarchical protection of the p-adic geometry. Base-invariant formulations therefore unify quantum error correction and the origin of dimensionless constants: the same tree that protects quantum information also gives rise to constants via projections independent of base.

6.

Syntactic Primitives and Base-Invariant Ontology

6.1 The Primacy

of Relations Over Numbers

The preceding analysis has revealed that base-invariant properties—continued fractions, valuations, scaling operators—are more fundamental than any particular numeral representation. This leads to a deeper ontological question: what is the minimal set of primitives needed to describe physical constants without anthropocentric bias? The answer, developed through the attached dialogues, is that **numbers themselves are emergent**; the true primitives are **syntactic relations** from which numerical values arise.

6.2 Core Syntactic

Primitives

Following the pure-syntax formulation (0.4.1.5.md), we posit four irreducible primitives:

Scaling relation $\prec_q$: a binary relation meaning “ y is one q -refinement of x ”. This replaces the notion of multiplication by a factor q .

Composition $\circ$: a partial operation that combines entities syntactically; it respects the scaling relations.

Distinction $\neq$: a primitive notion of difference, needed to avoid trivial collapse.

Coarse-graining rule $\rightarrow_M$: a rewrite rule that projects fine-grained syntactic structures onto coarse-grained descriptions (the Monna map).

These primitives contain no numbers, no sets, no predefined algebraic operations. All familiar mathematical objects—integers, reals, p-adic fields—emerge as equivalence classes of syntactic patterns.

6.3

Emergence of Integers, Primes, and Valuations

From the scaling relation $\prec_q$ we obtain the notion of **depth**: the equivalence class of entities connected by chains of $\prec_q$. If we designate a distinguished entity 0, the chain

$$0 \prec_q 1_q \prec_q 2_q \prec_q \dots$$

yields tokens n_q that behave as integers. Different scaling operators q give different families of integers, but those corresponding to **irreducible** scaling relations (those that cannot be decomposed as $\prec_{q_1} \circ \prec_{q_2}$) are the **prime scaling relations**. The integer primes $2, 3, 5, \dots$ are merely labels for the irreducible scaling relations that happen to produce finitely distinguishable tokens in human cognition.

This perspective reveals that **primes are not universal invariants** but convenient fictions of number theory, not a necessity of nature. In a fully democratic, unit-free system, primes are not fixed points of any non-trivial automorphism; they become fixed only if one artificially restricts the automorphism group to those that preserve the integer lattice—which reintroduces integer bias. Consequently, patterns such as the Riemann zeta zeros emerge from the discrete lattice $\mathbb{Z}$ and its multiplicative structure. Without integers, the zeta function has no Euler product, no primes, and no discrete zeros; the “prime pattern” dissolves into the continuous scaling symmetry of the positive reals.

The **valuation** $v_q(x)$ emerges as the maximal depth of common refinement: how many steps of $\prec_q$ are needed to reach a common ancestor. This is a purely syntactic notion that requires no real numbers.

6.4 The Monna

Map as a Coarse-Graining Rule

The Monna map M_q is an instance of the coarse-graining primitive $\rightarrow_M$. It deterministically projects infinite q -adic expansions (syntactic chains) onto bounded real intervals. The “loss” of information under this projection is not a flaw but the very mechanism that generates the appearance of continuity and real numbers. In physical terms, **measurement** is the application of $\rightarrow_M$ to the full syntactic state of the universe.

6.5 Implications

for Fundamental Constants

If constants like π , e , α^{-1} are not numbers but scaling operators, their values arise from the structure of the syntactic network. For example:

π is the scaling operator $\prec_\pi$ that appears as an irreducible relation in the cosmic syntax.

The fine-structure constant α emerges from the relative depths of electromagnetic scaling relations.

The proton-electron mass ratio μ reflects the nesting of different scaling operators (π , e , etc.) in the hadronic sector.

The continued-fraction expansions and q -adic valuations computed earlier are **syntactic fingerprints** of these operators, not arbitrary decimals.

6.6 Bias-Explicit

Universal Language

To avoid privileging any representation, we can formalize the above in a **parametric universal language** where every assumption (logic, algebraic structure, metric, computational model) is an explicit parameter. This language, outlined in the dialogue, allows comparisons across different biases and makes the hidden assumptions of conventional physics transparent. It is the ultimate expression of the Principle of Base Invariance.

6.7 Summary: From

Numbers to Syntax

The base-invariant investigation thus reaches its logical conclusion: the fundamental “source code” of the universe is not written in numbers but in **syntactic relations**. The patterns we have catalogued—large partial quotients, distinctive valuations, scaling symmetries—are clues to this deeper syntax. This perspective dissolves the anthropocentric prejudices of integer primes, real continua, and even fixed dimensions, offering a truly democratic framework for understanding physical constants.

7. Base-Free

Formulations of Physical Law

The **Principle of Base Invariance** demands that fundamental physical laws be expressible without privileging any numeral system. The adelic framework and q-adic scaling operations provide the mathematical tools to achieve this. In this section we outline how physical theories can be reformulated in a base-free manner, treating dimensionless constants not as mere numbers but as intrinsic scaling operators.

7.1 From Real

Numbers to Scaling Operators

Conventional physics expresses laws using real numbers, which are the completion of $\mathbb{Q}$ at the infinite prime ($\mathbb{R} = \mathbb{Q}_\infty$). The adelic perspective treats all completions— $\mathbb{R}$ and $\mathbb{Q}_p$ for each prime p —on equal footing. The q-adic generalization extends this to arbitrary scaling bases $q > 1$. In this framework, a dimensionless constant C is naturally associated with a scaling operator S_C acting on a q-adic space $\mathbb{Q}_q$, where q is chosen to be C itself (or a simple function of C). For example:

π -adic scaling: Rotational phenomena are described by fields on $\mathbb{Q}_\pi$, with π acting as the fundamental angular-to-linear scaling operator.

ϕ -adic scaling: Growth processes and self-similar structures are described on $\mathbb{Q}_\phi$, with ϕ as the fundamental growth-ratio operator.

e -adic scaling: Exponential decay and entropic processes are described on $\mathbb{Q}_e$, with e as the fundamental continuous-growth operator.

α^{-1} -adic scaling: Electromagnetic interactions are described on $\mathbb{Q}_{\alpha^{-1}}$, with $\alpha^{-1} \approx 137.036$ as the fundamental coupling-strength operator.

7.2 The Monna

Map as a Measurement Projection

Physical measurements yield real numbers, but the underlying dynamics may be ultrametric. The Monna map $M_q : \mathbb{Z}_q \rightarrow [0, 1]$ provides a canonical projection from the q-adic integers to the real interval. A base-free formulation keeps the dynamics in $\mathbb{Q}_q$ and applies the Monna map only at the stage of measurement. This is analogous to the

quantum-classical transition: the full quantum state lives in a high-dimensional Hilbert space, but measurements project onto classical observables.

7.3

Example: Maxwell's Equations in α^{-1} -Adic Form

Consider the fine-structure constant α . In a base-free formulation, electromagnetism is defined on an α^{-1} -adic space. The electric and magnetic fields are functions on the Bruhat–Tits tree $T_{\alpha^{-1}}$. Maxwell's equations become difference equations on the tree, with the coupling strength encoded in the branching structure. In the continuum limit (via the Monna map), one recovers the usual Maxwell equations with α appearing as a prefactor. The large partial quotient 277 in α^{-1} 's continued fraction may correspond to a special branching pattern that yields the observed value.

7.4

Example: Schrödinger Equation with π -Adic Scaling

For quantum systems with rotational symmetry, the natural scaling base is π . The Schrödinger equation can be written as a Vladimirov-type operator on $\mathbb{Q}_\pi$. The factor 2π that appears in the canonical commutation relation $[x, p] = i\hbar$ emerges from the Monna-map projection of a π -adic tree structure. The partial quotient 292 in π 's continued fraction may reflect a deep branching property that gives rise to the exceptional rational approximation 355/113.

7.5 Unification via the

Adele Ring

The adèle ring $\mathbb{A} = \mathbb{R} \times \prod_p \mathbb{Q}_p$ (generalized to include q -adic completions) offers a unified arena for all physical phenomena. Different forces may “live” in different completions: electromagnetism in $\mathbb{Q}_{\alpha^{-1}}$, strong interactions in $\mathbb{Q}_2$ (binary scaling), gravity in $\mathbb{R}$ (macroscopic continuum). The apparent independence of dimensionless constants could then be a consequence of their origin in distinct scaling sectors. Interactions between sectors are mediated by adelic reciprocity laws, which are number-theoretic analogues of coupling constants.

7.6 Experimental

Signatures

Base-free formulations make testable predictions: 1. **Time variation of constants:** If a constant is a topological invariant of a q -adic tree, its time variation is tightly constrained to discrete jumps corresponding to changes in branching patterns. 2. **Quantum gravity imprints:** At the Planck scale, spacetime may have an ultrametric structure, leaving signatures in the cosmic microwave background or gravitational wave spectra. 3. **High-precision metrology:** The continued fraction expansions of constants may reveal new rational approximations that could be searched for in atomic physics experiments.

7.7 Outlook

Base-free physics is not merely a reformulation; it is a paradigm shift that treats dimensionless constants as scaling operators rather than mere numbers. This perspective naturally incorporates number-theoretic patterns (large partial quotients, p -adic valuations) into the fabric of physical law. The ultimate goal is a complete adelic formulation of all fundamental interactions, where the “source code” of the universe is written in the language of valuations, continued fractions, and scaling operators.

8. Conclusion

We have conducted a systematic exploration of base-invariant number-theoretic patterns in fundamental constants. The most striking finding is the large partial quotient **277** in the continued fraction of the reciprocal fine-structure constant α^{-1} . This, together with the elevated geometric mean of its partial quotients, suggests that α^{-1} is not a generic real number but possesses special Diophantine properties. Similar patterns appear in the proton-electron mass ratio and other dimensionless constants, hinting at a common origin.

The **Principle of Base Invariance**—that fundamental laws should not depend on the choice of number representation—leads naturally to formulations in terms of continued fractions, p-adic valuations, and ultrametric geometry. The Monna map provides a concrete mechanism for projecting hierarchical p-adic structures onto the real numbers we measure, linking the discrete, fault-tolerant world of ultrametric quantum computation with the continuous, noisy world of conventional physics.

Speculatively, dimensionless constants may be **topological invariants** of an underlying p-adic tree, or **special values** of modular forms and periods. These ideas connect to deep threads in modern mathematics: motives, algebraic cycles, and the Langlands program. They also offer new perspectives on old puzzles: why $\alpha \approx 1/137$, why the proton-electron mass ratio is ~ 1836 , and why the cosmological constant is so small yet non-zero.

While many of these connections remain conjectural, they provide a rich framework for future research, both computational and theoretical. The search for a number-theoretic understanding of fundamental constants is not merely an academic exercise; it is a quest for the ultimate “source code” of the universe—a set of mathematical relations that are independent of any arbitrary representation, truly base-invariant.

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