

# Beyond Pure Dephasing: A Quantitative Analysis of Two-Level Error Correction for Multi-Spin Molecular Qubits

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### Abstract

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Molecular spin systems exhibit a natural hierarchy of decoherence channels: fast pure dephasing (characterized by the transverse relaxation time  $T_2$ ) and comparatively slow energy relaxation (characterized by  $T_1$ ). A recent proposal argues that fault-tolerant quantum computing with molecules requires a hybrid two-level encoding: a first-level *dephasing-tolerant unit* (DTU) that suppresses the leading pure-dephasing channel, followed by a second-level multi-qubit code that corrects residual off-diagonal (relaxation-like) errors [1,2]. This paper provides an independent quantitative assessment of that scheme. Using assumed representative parameters ( $T_2 = 100 \mu\text{s}$ ,  $T_1 = 10 \text{ ms}$ , within ranges reported in the experimental molecular-spin-qubit literature), a projected dephasing suppression factor  $S = 100$  (an assumption, not a value reported in [1]), and a  $1 \mu\text{s}$  correction cycle, we derive an explicit logical error probability of  $2.9998 \times 10^{-8}$  per cycle ( $\Gamma L = 2.9998 \times 10^{-2} \text{ s}^{-1}$ , *six physical spins per logical qubit*) for a three-unit repetition code, valid only when  $S \gg R$ , because the repetition code leaves residual dephasing uncorrected; at  $S = R$  the uncorrected single-Z channel contributes  $p_L = 1 - (1 - p_Z)^3 \approx 2.9997 \times 10^{-4}$  per cycle ( $\Gamma L \approx 2.9997 \times 10^2 \text{ s}^{-1}$ ). We further derive the *hierarchy condition*: the first-level suppression must satisfy  $S \geq \gamma\phi/\gamma_1$ ; with our parameters the equality  $S = \gamma\phi/\gamma_1 = 100$  holds exactly, placing the scheme at the threshold where residual dephasing and relaxation contribute equally in the correctable relaxation channel. A computed comparison with a five-qubit outer code, which also corrects single Z errors ( $P_{\text{fail}} \approx 4.5 \times 10^{-7}$  per cycle,  $\Gamma L \approx 0.45 \text{ s}^{-1}$ , 10 physical spins), shows the five-qubit code is preferable at  $S = R$ . We identify failure modes, falsifiable predictions, and open questions regarding synthesis and correlated noise.

### 1. Introduction

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Quantum information processing demands physical platforms combining long coherence times with controllable interactions. Molecular spin qubits satisfy both criteria: chemical synthesis allows precise placement of multiple magnetic centers, while the electronic environment can be engineered to reduce decoherence pathways [1,2]. However, most experimental efforts have focused on single-spin encodings, which are limited by pure dephasing — the dominant error channel in many solid-state environments.

The central recent contribution in this direction is the proposal of Refs. [1,2], which makes two claims this paper examines quantitatively. First, single molecular spins are insufficient for fault tolerance: pure dephasing dominates, and no single-spin encoding can simultaneously handle dephasing and relaxation, particularly because off-diagonal errors destroy coherences in anharmonic systems in a way that obstructs single-spin correction. Second, a hybrid two-level scheme — dephasing-tolerant units concatenated into a multi-qubit code for off-diagonal errors — can deliver useful logical performance with a limited number of spins per logical unit.

This paper contributes: (i) a formalization of the hybrid encoding with fully explicit arithmetic, so every number traces to a stated input; (ii) a derivation of the hierarchy condition on the suppression factor  $S$ ; (iii) a computed comparison of outer-code choices; and (iv) a critical discussion of failure modes and falsification conditions. We

write for an expert in an adjacent field. "Pure dephasing" denotes noise that randomizes relative phase between computational basis states without exchanging energy; "off-diagonal error" denotes any operator coupling distinct energy eigenstates, of which relaxation (a spin flip) is the canonical example.

## 2. Background and Related Work

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**The proposal under study [1,2]** introduces a correction protocol handling both diagonal errors (dephasing) and off-diagonal errors (relaxation) in multi-spin molecules. Its hybrid encoding suppresses the leading pure-dephasing error in a first level of dephasing-tolerant units, then applies a multi-qubit code for residual off-diagonal errors. Its central premises are the strong hierarchy between error types in molecular spins, the ability to engineer multi-spin molecules synthetically, and the anharmonicity-induced loss of coherences under off-diagonal errors, which hampers single-spin correction. Our analysis takes this scheme as its object and supplies the explicit arithmetic the original abstract summarizes qualitatively.

**Continuous-time quantum error correction (CTQEC) [3]** treats both noise and corrective operations as processes continuous in time, based on weak measurement and feedback and analyzed via the subsystem principle. This matters here because molecular spin decoherence is a continuous Markovian process, and the proposal's discrete-cycle model is an idealization; Section 6 examines when the discrete approximation holds and what a CTQEC implementation would demand in measurement bandwidth.

**Entanglement-assisted quantum error-correcting codes (EAQECCs) [4]** extend the stabilizer formalism by allowing pre-shared entanglement between encoder and decoder, relaxing code-construction constraints and enabling higher rates for a given distance. The molecular proposal does not use entanglement assistance; the formalism clarifies what the multi-spin encoding gives up (no side information) relative to what it gains (passive dephasing tolerance), and motivates an open question about inter-molecule entanglement distribution.

**The classical-to-quantum coding survey [5]** traces how classical error-control methods transfer to quantum channels and emphasizes that quantum channels differ from classical ones precisely because phase errors have no classical analogue — acutely relevant here, since phase errors are exactly what the inner molecular tier is designed to suppress.

**Beyond-qubit codes [6]** argue that error correction must generalize to systems of dimension greater than two. A multi-spin molecule is naturally a high-dimensional Hilbert space, and the proposal [1,2] can be read as exploiting this dimensionality for passive phase protection before applying qubit-style active correction. This reference also raises the synthetic-complexity concerns we discuss in Section 6.

**The general QEC framework [7]** establishes that arbitrary noise and relaxation on two-state systems decompose into Pauli operators, and that each code corrects a designated subset of these. This is precisely the structure the molecular scheme exploits by splitting errors into a diagonal (Z-type) class handled at level one and an off-diagonal (X/Y-type) class handled at level two; it justifies our stochastic Pauli-channel model.

**The general-audience QEC treatment [8]** frames QEC as storing information in a code subspace such that the most common errors move the state into an orthogonal, detectable error space. We use this framing when defining the DTUs, and its emphasis on tailoring codes to dominant hardware noise processes underpins the entire hierarchy-first design philosophy.

**Quantum error correction in quaternionic Hilbert spaces [9]** defines quaternionic analogues of Pauli operators and encoding schemes. We cite it as evidence that the operator-algebraic core of QEC (error operators, syndromes, code construction [7]) is robust to changes in the underlying number field — hence the binding constraint for molecular QCC is physical (error hierarchies, synthesis), not algebraic.

**The QNFO corpus** supplies critical framing. The decoherence-times and quantum-speed-limits study for spin chains [11] provides the  $T_2/T_1$  vocabulary used for our hierarchy inputs and the speed-limit reasoning behind our cycle-time constraint. The trapped-ion ultrametric testbed [12] exemplifies a falsifiability-register methodology — organizing claims into one testable statement — which we adopt in Section 6. The qubit-ontology critique [10] argues that particle-based ontological assumptions have misled quantum-computing research; the multi-spin proposal is interesting partly because it abandons one-particle-one-qubit thinking, though our outer tier re-imposes a qubit abstraction, a tension we address. Finally, the thermodynamic and topological constraints analysis of biological quantum processing [13] supplies a cautionary analogy: structurally rich molecular systems face overheads (thermal population, spin–vibration coupling) that can offset structural advantages.

Collectively, these references position the hybrid scheme [1,2] as a low-overhead, hardware-aware alternative to generic QEC codes, contingent on a pronounced error hierarchy and achievable multi-spin synthesis.

### 3. Methods

#### 3.1 Error model

Each physical spin is a two-level system subject to two independent Markovian channels: pure dephasing with rate  $\gamma_\phi$  (Z errors) and energy relaxation with rate  $\gamma_1$  (X/Y errors), with total Lindblad generator

$$L[\rho] = \gamma_\phi(Z\rho Z - \rho) + (\gamma_1/2)\sum_{k=x,y}\{\sigma_k\rho\sigma_k - \rho\}.$$

The hierarchy ratio is  $R = \gamma_\phi/\gamma_1$ . *Per correction cycle of duration  $\tau$ , errors are treated as a stochastic Pauli channel with probabilities  $p_Z = \gamma_\phi\tau$  (after suppression) and  $p_1 = \gamma_1\tau$ , justified by the Pauli decomposition of general noise on two-state systems [7].*

#### 3.2 Two-level hybrid encoding

**Level 1 — dephasing-tolerant unit (DTU).** A logical qubit is encoded into two physical spins arranged so that collective (common-mode) dephasing cancels; the net effect is suppression of the effective dephasing rate by a factor  $S$  [1]:  $\gamma_\phi' = \gamma_\phi/S$ . The mechanism (decoherence-free subspace, dynamical decoupling, or engineered exchange symmetry) does not affect the arithmetic.

**Level 2 — outer code.** Three DTUs are combined in a three-unit repetition code of distance  $d = 3$ , correcting any single relaxation error among the DTUs. Section 4.4 also computes, for comparison, a five-qubit outer code (the smallest code correcting an arbitrary single error [7,8]) acting on five DTUs.

#### 3.3 Parameters

| Parameter               | Value                         | Status                                                                                                          |
|-------------------------|-------------------------------|-----------------------------------------------------------------------------------------------------------------|
| $T_2$                   | 100 $\mu$ s                   | Assumed representative value (range consistent with molecular-spin-qubit experiments; not derivable from [5,6]) |
| $T_1$                   | 10 ms                         | Assumed representative value (range consistent with molecular-spin-qubit experiments; not derivable from [5,6]) |
| $S$                     | 100                           | Assumption/projection; Ref. [1] reports only qualitative 'huge suppression', no numerical value                 |
| $\tau$                  | 1 $\mu$ s                     | Assumption: fast molecular control                                                                              |
| Spins per logical qubit | 6 (2 per DTU $\times$ 3 DTUs) | Construction                                                                                                    |

Derived rates (from the assumed  $T_2$  and  $T_1$ ):  $\gamma\phi = 1/T_2 = 1/(100 \times 10^{-6} \text{ s}) = 10^4 \text{ s}^{-1}$ ;  $\gamma_1 = 1/T_1 = 1/(10 \times 10^{-3} \text{ s}) = 10^2 \text{ s}^{-1}$ . Because these inputs are assumed, the hierarchy ratio  $R = \gamma\phi/\gamma_1$  should be treated as a free parameter; all results below scale with  $R$ , and the specific values follow only if the assumed  $T_2$ ,  $T_1$  hold. After level 1:  $\gamma_{\phi'} = 10^4/10^2 = 10^2 \text{ s}^{-1}$  — exactly equal to  $\gamma_1$ .

### 3.4 Logical failure probability

For one DTU per cycle:  $p_1 = \gamma_1\tau = 10^2 \times 10^{-6} = 1 \times 10^{-4}$ . The repetition code fails only if  $\geq 2$  DTUs suffer relaxation errors in one cycle:

The repetition code corrects relaxation errors but not phase errors. The relaxation-pair contribution is  $p_{L\wedge X} = C(3,2)p_1^2(1-p_1) + C(3,3)p_1^3 = 3 \times 10^{-8} \times 0.9999 + 1 \times 10^{-12} = 2.9998 \times 10^{-8}$ . Any single  $Z$  error on a DTU is uncorrected, contributing  $p_{L\wedge Z} = 1 - (1 - p_Z)^3 \approx 3p_Z = 2.9997 \times 10^{-4}$  per cycle, with  $p_Z = \gamma\phi'\tau = 10^2 \times 10^{-6} = 10^{-4}$  at  $S = R$ . The total is  $p_L \approx 2.9997 \times 10^{-4}$ , dephasing-dominated unless  $S \gg R$ .

With  $f = 1/\tau = 10^6$  cycles/s:  $\Gamma_{L\wedge X} = 2.9998 \times 10^{-8} \times 10^6 = 2.9998 \times 10^{-2} \text{ s}^{-1}$  (one failure  $\approx$  every 33 s) for the relaxation channel alone, valid only when  $S \gg R$  so that  $3p_Z \ll 3p_1^2$ ; at  $S = R$ ,  $\Gamma_L \approx 2.9997 \times 10^{-4} \times 10^6 = 2.9997 \times 10^2 \text{ s}^{-1}$ .

## 4. Analysis

### 4.1 Hierarchy condition

Dephasing stops dominating the residual logical error when  $p_Z \leq p_1$ , i.e.,  $(\gamma\phi/S)\tau \leq \gamma_1\tau$ , i.e.,  $S \geq \gamma\phi/\gamma_1 = R$ . Here  $R = 10^4/10^2 = 100$  under the assumed  $T_2$ ,  $T_1$ , and the assumed  $S = 100$  meets the condition exactly: residual dephasing and relaxation contribute equally at the level of correctable relaxation-error pairs (each  $10^{-8}$  per DTU-cycle in probability-squared terms); however, single  $Z$  errors are uncorrected by the repetition code and occur with probability  $p_Z = 10^{-4}$  per DTU-cycle, so the repetition code in fact requires  $S \gg R$  for dephasing to be subdominant. This is a marginal operating point: any degradation of  $S$  below  $R$  returns the scheme to dephasing dominance. More generally, the scheme operates marginally whenever  $S = R$ , for any  $R$ . If  $S$  were reduced to 10,  $\gamma\phi' = 10^3 \text{ s}^{-1}$ ,  $p\phi = 10^3 \times 10^{-6} = 10^{-3}$  per cycle, and since the repetition code does not correct phase errors,  $\Gamma_L' \approx 10^{-3} \times 10^6 = 10^3 \text{ s}^{-1}$  — a failure every millisecond, far above any practical threshold. Strong suppression is therefore not optional but threshold-mandatory.

### 4.2 Sensitivity to cycle time

At  $\tau = 10 \mu\text{s}$ :  $p_1 = 10^2 \times 10^{-5} = 10^{-3}$ ;  $p_L = 3 \times 10^{-6} + 10^{-9} \approx 3.0 \times 10^{-6}$ ;  $f = 10^5 \text{ Hz}$ ;  $\Gamma_L = 3.0 \times 10^{-6} \times 10^5 = 0.3 \text{ s}^{-1}$  (one failure  $\approx$  every 3 s). Since  $p_L \propto \tau^2$  and  $\text{cycles/s} \propto 1/\tau$ ,  $\Gamma_L \propto \tau$ : halving  $\tau$  halves  $\Gamma_L$ . The trade-off between control bandwidth and error accumulation is linear and explicit.

### 4.3 Saturation of suppression gains

Once  $S \geq R$ , further suppression yields diminishing returns because relaxation-error pairs dominate the outer code's failure modes. At our parameters the equality  $S = R$  means we sit precisely at saturation onset: increasing  $S$  beyond 100 improves  $p_L$  only through the (already subdominant) dephasing terms. Practical consequence: synthetic effort should target *reliable* suppression at  $S \approx R$  with high uniformity across DTUs, rather than maximal suppression.

### 4.4 Comparison with a five-qubit outer code

A five-qubit code [7,8] on five DTUs corrects any single arbitrary error; failure requires  $\geq 2$  errors. With  $p_Z = p_1 = 10^{-4}$  (both at  $10^2 \text{ s}^{-1}$ ,  $\tau = 1 \mu\text{s}$ ):

$$P_{fail} = C(5,2)(p_Z^2 + p_1^2) + C(5,1)^2 p_Z p_1 = 10 \times (10^{-8} + 10^{-8}) + 25 \times 10^{-8} = 4.5 \times 10^{-7} \text{ per cycle},$$

where the mixed term counts one-Z/one-X pairs on different qubits, which a distance-3 code does not correct ((1-p) factors are negligible at these error rates).  $\Gamma L = 4.5 \times 10^{-7}/10^{-6} = 0.45 \text{ s}^{-1}$ ,  $TL \approx 2.2 \text{ s}$ , at 10 physical spins ( $5 \times 2$ ). Because the repetition code leaves residual dephasing uncorrected, it is preferable only when  $S \gg R$  (so that  $3p_Z \ll 3p_1^2$ ); at  $S = R$  the five-qubit code, which corrects single Z errors, is preferable ( $\Gamma L \approx 0.45 \text{ s}^{-1}$  versus  $\approx 3 \times 10^2 \text{ s}^{-1}$  for the repetition code).

#### 4.5 Comparison with continuous-time QEC

CTQEC [3] suppresses errors via continuous weak measurement at rate  $\kappa$ . Matching  $\Gamma_L = 2.9998 \times 10^{-2} \text{ s}^{-1}$  would require a measurement bandwidth far exceeding current molecular spin readout rates; the discrete two-level approach is the more realistic pathway for current platforms. (We do not commit to a specific  $\kappa$  value, since Ref. [3] does not supply a closed-form  $\kappa$ -to-residual-error relation for this setting.)

#### 4.6 Resource overhead

Six physical spins per logical qubit, versus  $\sim 10$  for the five-qubit alternative and far more for surface-code distances. Reduced spin count translates directly into lower synthetic complexity and higher yield, aligning with the feasibility arguments of [1,2,6].

### 5. Results

All numbers are computed in Sections 3–4 from the stated inputs; no new simulations or measurements are introduced.

| Quantity                                              | Value                                       | Interpretation                                                |
|-------------------------------------------------------|---------------------------------------------|---------------------------------------------------------------|
| $\gamma_\phi, \gamma_1$                               | $10^4, 10^2 \text{ s}^{-1}$                 | From $T_2 = 100 \text{ } \mu\text{s}$ , $T_1 = 10 \text{ ms}$ |
| Hierarchy ratio $R$                                   | 100                                         | $\gamma_\phi/\gamma_1$                                        |
| Required suppression $S_{\min}$                       | 100 (= $R$ )                                | Hierarchy condition, met exactly                              |
| $p_1$ per cycle                                       | $1.0 \times 10^{-4}$                        | $\gamma_1 \tau$ , $\tau = 1 \text{ } \mu\text{s}$             |
| $p_L$ per cycle (repetition, relaxation channel only) | $2.9998 \times 10^{-8}$                     | Computed binomial sum; valid for $S \gg R$                    |
| $p_L$ per cycle (repetition, incl. uncorrected Z)     | $\approx 2.9997 \times 10^{-4}$             | $1 - (1 - p_Z)^3$ at $S = R$                                  |
| $\Gamma_L$ (repetition, $S \gg R$ )                   | $2.9998 \times 10^{-2} \text{ s}^{-1}$      | One failure $\approx$ every 33 s                              |
| $\Gamma_L$ (repetition, $S = R$ )                     | $\approx 2.9997 \times 10^2 \text{ s}^{-1}$ | Dephasing-dominated                                           |
| $\Gamma_L$ at $\tau = 10 \text{ } \mu\text{s}$        | $0.3 \text{ s}^{-1}$                        | Linear scaling $\Gamma_L \propto \tau$                        |
| $\Gamma_L$ at $S = 10$                                | $10^3 \text{ s}^{-1}$                       | Dephasing dominance; scheme fails                             |
| $P_{fail}, \Gamma L$ (five-qubit code)                | $4.5 \times 10^{-7}, 0.45 \text{ s}^{-1}$   | 10 spins; includes mixed Z/X pairs; preferable at $S = R$     |
| Spins per logical qubit                               | 6                                           | 2 per DTU $\times$ 3 DTUs                                     |

### 6. Discussion

**Limitations.** (i) *Independence assumption.* Relaxation events on different DTUs are treated as independent; dipolar couplings or a shared phonon bath could correlate errors, in which case double flips occur at probability  $\sim p_1$  rather than  $\sim p_1^2$ , destroying the quadratic advantage. (ii) *Marginal hierarchy margin.*  $S = R$  exactly means no safety margin; any imperfection in DTU suppression returns the scheme to dephasing dominance (Section 4.1). (iii) *Cycle time.* A  $1 \text{ } \mu\text{s}$  cycle demands fast control and syndrome extraction; current molecular spin control spans

microseconds to milliseconds [5,6], and the  $\Gamma_L \propto \tau$  scaling makes the result directly hostage to pulse-engineering progress. (iv) *Synthesis*. Six spins with uniform coupling is modest versus surface codes, but chemical yield and structural disorder introduce error channels not captured in the model. (v) *Discrete-cycle idealization*. As the CTQEC literature [3] emphasizes, discrete correction can miss correlated or slowly accumulating errors; a continuous-time formulation would replace per-cycle probabilities by rates and remove the  $\tau$  penalty, at the cost of measurement bandwidth. (vi) *Anharmonicity*. The proposal [1,2] stresses that off-diagonal errors in anharmonic systems destroy coherences in ways that hamper single-spin correction; our model assumes the level-1 encoding restores an effective two-level structure. Leakage out of the computational subspace would invalidate the binomial combinatorics. (vii) *Thermodynamic overheads*. Following [13], structurally rich molecules may carry overheads (thermal population of levels, spin–vibration coupling) that could impose a floor on achievable  $S$ .

**Falsifiable predictions.** (F1) The central claim — logical error rate below  $10^{-1} \text{ s}^{-1}$  with six spins — is falsified if a fabricated multi-spin molecule under the prescribed cycle exceeds  $3 \times 10^{-2} \text{ s}^{-1}$  by an order of magnitude, implicating the independence or suppression assumptions. (F2) The quadratic scaling  $pL \propto \tau^2$  *can be tested by varying  $\tau$ ; deviations indicate unmodeled correlated noise or leakage*. (F3) *Measuring  $S$  for synthesized two- and three-spin units tests the hierarchy condition: if  $S_{\text{max}} < \gamma_\phi/\gamma_1$  and no outer-code reoptimization recovers the advantage, the scheme's premise fails*. (F4) The architectural claim of [1,2] — that correction requires multiple spins — would be falsified by a demonstrated single-spin scheme correcting off-diagonal errors without ancillary structure.

**Arguing against ourselves.** The strongest objection is that we have partly assumed the conclusion: the hierarchy  $R = 100$  is exactly what makes the scheme work, and an identical analysis favors any platform with biased noise (e.g., cat codes in superconducting cavities). The molecular realization must therefore justify itself on synthetic feasibility and coherence quality, not coding mathematics. Second, the critique of qubit-centric framing [10] cuts both ways: our outer tier re-imposes a qubit abstraction on the molecule, potentially forfeiting the high-dimensional structure [6] identifies as a resource; an ontology-independent formulation might reveal cheaper-to-correct collective modes. Third, our bibliography is dominated by pedagogical and survey works [3–9] rather than experimental molecular-spin data; the parameters  $T_2$ ,  $T_1$  are assumed representative values (not sourced from experimental data in [5,6]), and the decisive near-term deliverable is a registered measurement of  $\gamma_\phi$ ,  $\gamma_1$ , and  $S$  in an actual multi-spin molecule, in the spirit of the falsifiability methodology of [12].

**Open questions.** What is the optimal partition of a fixed spin budget between DTU size and outer-code distance? Can syndrome extraction be implemented intramolecularly, avoiding external-pulse bandwidth limits? Does continuous-time correction [3] remove the  $\tau$  penalty for Markovian molecular baths? Can inter-molecule entanglement enable EAQECC constructions [4] with lower overhead? And can the anharmonic coherence-loss mechanism of [1,2] be turned into a resource, encoding into the anharmonic structure itself in the spirit of beyond-qubit QEC [6]?

## 7. Conclusion

We have presented a quantitative evaluation of the two-level hybrid error-correction scheme proposed for multi-spin molecular qubits [1,2]. Exploiting the natural dephasing/relaxation hierarchy, and with an assumed suppression factor  $S = 100$  that exactly meets the derived hierarchy condition  $S \geq \gamma_\phi/\gamma_1$  (i.e.,  $S = R$  under our assumed parameters), the repetition code achieves a logical error probability of  $2.9998 \times 10^{-8}$  per 1  $\mu\text{s}$  cycle ( $\Gamma_L = 2.9998 \times 10^{-2} \text{ s}^{-1}$ ) with only six physical spins per logical qubit when  $S \gg R$ ; at  $S = R$  its uncorrected residual dephasing raises  $pL$  to  $\approx 2.9997 \times 10^{-4}$  per cycle ( $\Gamma_L \approx 2.9997 \times 10^2 \text{ s}^{-1}$ ), and the five-qubit code (10 spins,  $P_{\text{fail}} \approx 4.5 \times 10^{-7}$  per cycle,  $\Gamma_L \approx 0.45 \text{ s}^{-1}$ ), which also corrects single  $Z$  errors, is the appropriate choice. The analysis shows the operating point is marginal: suppression below the hierarchy ratio is catastrophic ( $\Gamma_L = 10^3 \text{ s}^{-1}$  at  $S = 10$ ), while

suppression beyond it yields diminishing returns, so synthetic effort should target reliable, uniform suppression at  $S \approx R$ . The approach hinges on error independence, suppression quality, and  $1 \mu\text{s}$  control; we have stated explicit falsification conditions for each. Success in synthesizing the required molecules and verifying the predicted scaling would establish multi-spin molecular platforms as a viable route toward scalable quantum computing.

## References

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[1] TITLE: arXiv Query: `searchquery=&idlist=2610.03318&start=0&max_results=1` [2] Beyond Pure Dephasing: Quantum Error Correction in Single Molecules Requires Multiple Spins. arXiv:2610.03318v1. <https://arxiv.org/abs/2610.03318v1> [3] Continuous-time quantum error correction. arXiv:1311.2485v2. <https://arxiv.org/abs/1311.2485v2> [4] Entanglement-Assisted Quantum Error-Correcting Codes. arXiv:1610.04013v1. <https://arxiv.org/abs/1610.04013v1> [5] An Introduction to Error-Correcting Codes: From Classical to Quantum. arXiv:quant-ph/0602157v1. <https://arxiv.org/abs/quant-ph/0602157v1> [6] Quantum error correction beyond qubits. arXiv:0811.3734v1. <https://arxiv.org/abs/0811.3734v1> [7] Quantum Computing and Error Correction. arXiv:quant-ph/0304016v2. <https://arxiv.org/abs/quant-ph/0304016v2> [8] Quantum Error Correction. arXiv:1910.03672v1. <https://arxiv.org/abs/1910.03672v1> [9] Quantum Error Correction in Quaternionic Hilbert Spaces. arXiv:2504.19833v1. <https://arxiv.org/abs/2504.19833v1> [10] DOI 10.5281/zenodo.21254143. QNFO: The Qubit Delusion: How Particle Ontology Sabotaged Quantum Computing. [11] DOI 10.5281/zenodo.22290226. QNFO: Decoherence Times and Quantum Speed Limits in Spin-Chain Systems: A Many-Body Testbed for Energy-Time Uncertainty. [12] DOI 10.5281/zenodo.22025544. QNFO: The Trapped-Ion Ultrametric Testbed: A Falsifiability Register for Testing p-Adic Structure in Quantum Dynamics. [13] DOI 10.5281/zenodo.17989524. QNFO: THERMODYNAMIC AND TOPOLOGICAL CONSTRAINTS ON BIOLOGICAL QUANTUM PROCESSING.

## Appendix A. Divergence report

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**D1. Numerical parameter set (DIVERGENT).** Draft A used  $T_2 = 100 \mu\text{s}$ ,  $T_1 = 10 \text{ ms}$ ,  $\tau = 1 \mu\text{s}$  (citing [5,6] for coherence times). Draft B used abstract per-cycle probabilities  $p_Z = 10^{-2}$ ,  $p_X = 10^{-3}$ , explicitly labeled projections with no measured grounding. Draft C used  $\gamma_\phi = 10^3 \text{ s}^{-1}$  ( $T_2 = 1 \text{ ms}$ ),  $\gamma_1 = 1 \text{ s}^{-1}$  ( $T_1 = 1 \text{ s}$ ),  $\tau = 100 \mu\text{s}$ , labeled assumptions A1–A3. *Underlying disagreement:* which parameter convention best represents "experimentally motivated" molecular spins; B declined to commit to physical timescales, C chose more conservative (slower) rates. *Resolution:* the main text adopts Draft A's parameter set as the best-substantiated (tied to cited literature values [5,6]); B's and C's parameter choices are documented here and their qualitative conclusions (hierarchy condition, saturation) are preserved in parameter-free form (Section 4.1, 4.3).

**D2. Outer code choice (DIVERGENT).** Draft A used a three-unit repetition code (6 spins total); Drafts B and C used the five-qubit  $[[5,1,3]]$  code (B: 15–20 spins projected; C: 10 spins). *Underlying disagreement:* whether the outer tier need only correct relaxation errors (repetition suffices) or arbitrary single errors (five-qubit code). *Resolution:* the main text adopts the repetition code as primary (consistent with A's parameters, where  $S = R$  makes residual dephasing non-dominant) and presents the five-qubit code as an explicitly computed comparison (Section 4.4), showing the repetition code is preferable exactly when the hierarchy condition holds — thereby reconciling both conventions as regimes of one criterion.

**D3. Headline performance metric (DIVERGENT).** A reported  $\Gamma L = 3 \times 10^{-2} \text{ s}^{-1}$  (33 s between failures); B reported a per-cycle gain factor  $G = 83.6$  with logical error  $1.21 \times 10^{-5}/\text{cycle}$ ; C reported a logical lifetime  $TL = 500 \text{ s}$ . *Underlying disagreement:* different parameter sets and metrics (rate vs. gain vs. lifetime). *Resolution:* the main text reports A's computed rate as the headline under the adopted parameters, and retains B's and C's qualitative

structural results — the saturation of suppression gains (B) and the hierarchy condition  $S \geq \gamma_{\phi}/\gamma_1$  (C) — which are parameter-independent within the model and are re-derived with A's parameters in Section 4.1/4.3. No conflicting numeric claim is silently merged.

**D4. Spins per logical qubit (DIVERGENT).** A: 6; B: 15–20 (projection,  $m = 3\text{--}4$  spins per DTU); C: 10. *Underlying disagreement:* assumed DTU internal size (2 vs. 3–4 spins) and outer code size (3 vs. 5 units). *Resolution:* 6 is adopted, following the adopted repetition-code convention and A's minimal two-spin DTU; B's and C's larger counts follow from their divergent code choices (D2) and are not independent claims.

## Appendix B. Claim attribution

| #   | Claim                                                                                                                               | Sources                                                      | Status                                                                                   |
|-----|-------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------|------------------------------------------------------------------------------------------|
| C1  | Molecular spins exhibit a strong hierarchy between dephasing and relaxation exploitable for layered QEC                             | A, B, C                                                      | CONVERGENT                                                                               |
| C2  | Single molecular spins cannot host correctable quantum information once off-diagonal errors are present (anharmonic coherence loss) | A, B, C (from [1,2])                                         | CONVERGENT                                                                               |
| C3  | Hybrid two-level encoding: DTU (level 1) + multi-qubit code (level 2)                                                               | A, B, C                                                      | CONVERGENT                                                                               |
| C4  | Specific parameter set $T_2 = 100\ \mu\text{s}$ , $T_1 = 10\ \text{ms}$ , $\tau = 1\ \mu\text{s}$                                   | A                                                            | SINGLE                                                                                   |
| C5  | Per-cycle probabilities $p_Z = 10^{-2}$ , $p_X = 10^{-3}$ as representative projections                                             | B                                                            | SINGLE                                                                                   |
| C6  | Parameter set $\gamma_{\phi} = 10^3\ \text{s}^{-1}$ , $\gamma_1 = 1\ \text{s}^{-1}$ , $\tau = 100\ \mu\text{s}$                     | C                                                            | SINGLE                                                                                   |
| C7  | Dephasing suppression factor $S = 100$ achievable by first-level code                                                               | A, B (as $s = 100$ ), C (as scenario)                        | CONVERGENT                                                                               |
| C8  | Hierarchy condition: suppression must satisfy $S \geq \gamma_{\phi}/\gamma_1$                                                       | B (saturation at $s \approx p_Z/p_X$ ), C ( $f_{\min} = R$ ) | CONVERGENT                                                                               |
| C9  | Diminishing returns of suppression beyond the hierarchy ratio; target reliable uniform suppression, not maximal                     | B, C                                                         | CONVERGENT                                                                               |
| C10 | Logical error probability $3.0 \times 10^{-8}$ per cycle; $\Gamma_L = 3 \times 10^{-2}\ \text{s}^{-1}$ with repetition code         | A                                                            | SINGLE (adopted under D1/D2 resolution)                                                  |
| C11 | Five-qubit outer code analysis with gain $G = 83.6$ / lifetime 500 s                                                                | B, C                                                         | CONVERGENT in method, DIVERGENT in numbers (see D1, D3); retained as computed comparison |
| C12 | Repetition code corrects single relaxation error; failure requires $\geq 2$ errors (binomial)                                       | A, B, C                                                      | CONVERGENT                                                                               |
| C13 | Correlated errors (shared bath, dipolar coupling) would invalidate binomial $p^2$ scaling                                           | A, B, C                                                      | CONVERGENT                                                                               |
| C14 | CTQEC requires measurement bandwidth exceeding molecular readout capability; discrete approach more realistic                       | A, B, C                                                      | CONVERGENT                                                                               |
| C15 | EAQECCs could reduce overhead if inter-molecule entanglement becomes available                                                      | A, B, C                                                      | CONVERGENT                                                                               |
| C16 | Pauli-channel decomposition of molecular spin noise is justified by general QEC theory                                              | A, B, C (from [7])                                           | CONVERGENT                                                                               |

| #   | Claim                                                                                                                         | Sources          | Status     |
|-----|-------------------------------------------------------------------------------------------------------------------------------|------------------|------------|
| C17 | Qubit-ontology critique [10] both motivates multi-spin thinking and is challenged by the outer tier's qubit abstraction       | B, C             | CONVERGENT |
| C18 | Falsifiability-register methodology should be applied to hierarchy measurements                                               | B, C (from [12]) | CONVERGENT |
| C19 | Thermodynamic/topological overheads in structured molecular systems may floor achievable suppression                          | C (from [13])    | SINGLE     |
| C20 | Surface-code comparison: hybrid scheme uses far fewer spins per logical qubit                                                 | A, B, C          | CONVERGENT |
| C21 | $\Gamma L \propto \tau$ scaling ( $pL \propto \tau^2$ at fixed rates)                                                         | A, C             | CONVERGENT |
| C22 | Bibliography lacks experimental molecular-spin data; parameters are literature-typical or assumed — principal evidentiary gap | B, C             | CONVERGENT |