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Braid Group Representations and Modular Data: Non-Uniqueness, Finite Images, and the Ising Test Case

Rowan Brad Quni-Gudzin

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ABSTRACT

A central question in the theory of topological order asks whether the braid group representation B_N carried by N non-Abelian anyons uniquely determines the modular data — the S and T matrices — of the underlying topological quantum field theory. We reconcile three lines of analysis into a single assessment. Structurally, we formalize the question as a reduction map Φ from modular tensor category data to projective braid group representations, whose kernel is the locus of potential counterexamples. Two rigorous finiteness results constrain the problem: braid representations from twisted quantum doubles of finite groups factor through finite groups, and the $SO(N)_2/O(N)_2$ braid representations are Gaussian with finite image. We then perform an explicit computation on the Ising anyon model — the canonical non-Abelian order of Majorana zero modes — reconstructing the topological spins $\theta_\sigma = e^{i\pi/8}$, $\theta_\psi = -1$, the total quantum dimension $\mathcal{D} = 2$, and the full S matrix from the braid eigenvalues, with all arithmetic shown. [...]

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Abstract

A central question in the theory of topological order asks whether the braid group representation B_N carried by N non-Abelian anyons uniquely determines the modular data — the S and T matrices — of the underlying topological quantum field theory. We reconcile three lines of analysis into a single assessment. Structurally, we formalize the question as a reduction map Φ from modular tensor category data to projective braid group representations, whose kernel is the locus of potential counterexamples. Two rigorous finiteness results constrain the problem: braid representations from twisted quantum doubles of finite groups factor through finite groups, and the $SO(N)_2/O(N)_2$ braid representations are Gaussian with finite image. We then perform an explicit computation on the Ising anyon model — the canonical non-Abelian order of Majorana zero modes — reconstructing the topological spins $\theta_\sigma = e^{i\pi/8}$, $\theta_\psi = -1$, the total quantum dimension $\mathcal{D} = 2$, and the full S matrix from the braid eigenvalues, with all arithmetic shown. We exhibit a Galois-conjugate deformation Ising(7) with identical fusion rules but distinct S matrix, and verify that its braid representation is inequivalent to the Ising representation on B_3 by an explicit trace computation, while remaining indistinguishable at the level of fusion rules. We conclude that braid representations constrain but do not provably uniquely determine modular data in general: uniqueness can hold only under additional hypotheses, and the Majorana fusion-rule classification is not closed by braid data alone. The affirmative claim that braiding is a complete invariant is restricted to a narrower regime than sometimes asserted, and we document this divergence explicitly.

1. Introduction

Non-Abelian anyons in two spatial dimensions are characterized by two layers of algebraic structure. The first is combinatorial: fusion rules N_{ab}^c specifying how anyon charges combine. The second is geometric: adiabatic exchange of anyons induces unitary transformations on the degenerate fusion space, giving a representation of the braid group

B_N , generated by exchanges $\sigma_1, \dots, \sigma_{N-1}$ subject to $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ and $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$. The full topological order is encoded in the modular tensor category (MTC): simple objects, F -symbols, R -symbols, and the derived modular data (S, T) , which furnish a projective representation of $SL(2, \mathbb{Z})$.

The inverse question — posed explicitly in prior work [12] — is whether the braid representation $\rho_N : B_N \rightarrow U(\mathcal{H}_N)$ uniquely determines (S, T) . The stakes are practical: in Majorana platforms, braiding is the operationally accessible content of the anyon model [10,11]. If braiding data determined modular data uniquely, interferometric or braiding measurements would constitute complete tomography of the topological order. If not, additional observables (total quantum dimension, chiral central charge, ground-state degeneracy) are required.

This paper makes four contributions: (i) a formal statement of the uniqueness question as a map on fusion data and identification of its kernel; (ii) assembly of the localization and finiteness results that restrict where uniqueness can hold; (iii) a complete, arithmetically explicit reconstruction of the Ising modular data from braid eigenvalues, including the trace check $\text{Tr } \rho_3(\sigma_1) = \sqrt{2} e^{i\pi/8}$; (iv) an explicit Galois-conjugate deformation of the Ising theory with identical fusion rules but distinct modular data, together with an honest assessment of what the braid representation does and does not distinguish.

The paper is organized as follows. Section 2 surveys the literature. Section 3 sets up the formalism. Section 4 contains the derivations. Section 5 reports results. Section 6 discusses limitations and falsifiability. Section 7 concludes.

2. Background and Related Work

Localization of braid group representations. Rowell [1] initiated the systematic study of unitary braid group representations arising from unitary R -matrices and simple objects in unitary braided fusion categories, with particular attention to when such representations have finite image — a property that severely constrains admissible anyon models and that we exploit throughout. The generalization to (quasi-)localizations by Rowell and collaborators [7] determines when a family of braid representations can be uniformly modeled on tensor powers of a fixed vector space with generators acting locally. Localization is precisely the hypothesis under which braid data could reconstruct a category; as we show in Section 4, even granting localization, reconstruction is at best non-unique in general.

Finite-group constructions. Vafadar and collaborators [2] proved that braid group representations from twisted quantum doubles of finite groups always factor through finite groups — in sharp contrast to quantum groups at roots of unity, whose images are typically infinite. This supplies a large class of topological orders whose braid data is "classical" and cannot directly encode the full modular structure. A complementary unification via iterated twisted tensor products appears in [6], which hints that the braidings on G -gaugings of a pointed modular category $\mathcal{C}(A, Q)$ relate to those of $\mathcal{C}(A, Q)$ itself — multiple braidings over one fusion ring, foreshadowing non-uniqueness.

The $SO(N)_2$ family. Rowell, Zhang, Wu, and collaborators [4] described the centralizer algebras of tensor powers of spin objects in the pre-modular categories $SO(N)_2$ (N odd) and $O(N)_2$ (N even) in terms of quantum tori, and proved the corresponding braid group representations are Gaussian with finite image. Since $SO(3)_2$ is equivalent to the Ising category, this result directly governs the Majorana sector analyzed below. A finite braid image has finitely many matrix entries, so the image alone cannot witness arbitrarily rich modular data.

Geometric and holographic constructions. Bellingeri and Guaschi [3] constructed infinite families of nongeometric braid group representations via d -fold branched coverings, expressed as explicit actions on free-group generators — demonstrating that the space of braid representations vastly exceeds the space realized by anyons, an asymmetry at the conceptual heart of the non-uniqueness problem. The AdS/CFT analysis of [5] connects braid group representations to defect operators, mapping bulk Wilson loops to boundary defect operators via fusion and braiding in MTCs, illustrating that modular data carries information braid representations alone do not obviously encode.

Physics of anyons and Majorana platforms. The path-integral treatment of spinning anyons and the fractional quantum Hall effect [8] introduced charged winding numbers and super-Knizhnik–Zamolodchikov operators, showing how braid phases encode topological spin — the diagonal of T — but only up to the ambiguities quantified in Section 4. The metaplectic anyon models of Bombin and collaborators [9], combining an Ising sector with $SO(m)_2$ Chern–Simons theory, provide a complete account of quasiparticle types, fusion rules, and braiding for a family of models; we use the $m = 1$ (Ising) member as our worked example. Pedagogical and platform-focused treatments of Majorana zero modes in Kitaev chains [10] and fermion-parity-based computation [11] establish that the braid representation is the operationally accessible datum — and that measurement-based schemes inject non-braid resources (parity, measurement) outside ρ_N itself.

Program lineage. Reference [12] (QNFO, <https://papers.qnfo.org/papers/braid-group-representations-modular-data-and-the-classification-of-majorana-zero/>) posed the uniqueness question directly and initiated the MZM fusion-rule classification program; this paper is a re-entry from that work. Reference [13] (QNFO, <https://papers.qnfo.org/papers/majorana-zero-mode-parity-the-hexagon-equation-and-chiral-edge-modes-in-2d-topol/>) analyzed uniqueness of the exchange matrix from MZM parity operators and the hexagon equation, finding that parity data constrain but do not fix the full anyonic exchange matrix. Reference [14] (QNFO, "Ultrametric Relaxation Dynamics in Topological Quantum Memory") supplies the memory-relaxation context in which braid-image finiteness becomes an error model rather than an abstraction; static modular data are robust against such dynamics, justifying the purely algebraic focus here.

3. Methods

3.1 Categorical setup. A unitary modular tensor category $\mathcal{C}$ has simple objects $a, b, \dots$, fusion coefficients $N_{ab}^c \in \mathbb{Z}_{\geq 0}$, quantum dimensions d_a satisfying $d_a d_b = \sum_c N_{ab}^c d_c$, total quantum dimension $\mathcal{D} = \sqrt{\sum_a d_a^2}$, topological spins $\theta_a \in U(1)$ (diagonal of T), and an S matrix. We adopt the convention $S_{0a} = d_a/\mathcal{D}$ and the mutual-braiding form

$$S_{ab} = \frac{1}{\mathcal{D}} \sum_c N_{ab}^c \frac{\theta_c}{\theta_a \theta_b} d_c.$$

3.2 Braid representation. For N anyons of type a , the fusion space V_{a^N} carries $\rho_N : B_N \rightarrow U(V_{a^N})$; on a fusion path whose intermediate charge at the i -th exchange is x , the generator σ_i acts as $R_c^{x,x}$ composed with F -moves. The eigenvalue of σ_i on a channel where the two exchanged anyons fuse to c is R_c^{aa} . The ribbon identity in a multiplicity-free, gauge-fixed basis reads $\theta_c = \theta_a \theta_b (R_c^{ab})^2$, equivalently $R_c^{ab} = \sqrt{\theta_c/(\theta_a \theta_b)}$ with the hexagon equations fixing the branch of the square root.

3.3 The uniqueness map. Define

$$\Phi : \{\text{MTC data } (\mathcal{C}, S, T)\} \longrightarrow \{\text{projective representations } B_N \rightarrow PU(\mathcal{H}_N)\}.$$

The uniqueness conjecture is injectivity of Φ ; a counterexample is a pair of distinct MTCs with equivalent projective braid images. Two UMTCs with the same fusion ring are *braid-equivalent* if their representations intertwine for all N , up to an overall Abelian character $B_N \rightarrow U(1)$.

3.4 Finite-image test. If $\Gamma_N := \rho_N(B_N)$ is finite of order g_N , the braid matrices take finitely many values; two MTCs whose images coincide on the finite generating set are braid-indistinguishable at that N . For the Majorana (Ising) sector, the fusion space for N σ anyons with fixed total charge has dimension $2^{N/2-1}$ for even N (derived in Section 4.5), and the image is a finite Clifford-type group; we report order-of-magnitude bounds with stated assumptions.

3.5 Arithmetic protocol. Every quantitative claim below is computed with shown arithmetic; inputs are labeled with their sources. No measured or simulated data are used.

4. Analysis

4.1 The Ising anyon model. The Ising UMTC has simple objects $\{1, \sigma, \psi\}$ with fusion rules

$$\sigma \times \sigma = 1 + \psi, \quad \sigma \times \psi = \sigma, \quad \psi \times \psi = 1,$$

multiplicity-free. This is the fusion ring of Majorana zero modes: N σ anyons realize N MZMs with total fermion parity fixed [10,13].

4.2 Quantum dimensions. From $\sigma \times \sigma = 1 + \psi$ and $d_\psi = 1$ (invertible, $\psi \times \psi = 1$):

$$d_\sigma^2 = d_1 + d_\psi = 1 + 1 = 2 \Rightarrow d_\sigma = \sqrt{2}.$$

Total quantum dimension:

$$\mathcal{D} = \sqrt{d_1^2 + d_\sigma^2 + d_\psi^2} = \sqrt{1 + 2 + 1} = \sqrt{4} = 2.$$

4.3 Braid eigenvalues. The Ising R -symbols (standard gauge) are

$$R_1^{\sigma\sigma} = e^{-i\pi/8}, \quad R_\psi^{\sigma\sigma} = e^{3i\pi/8}, \quad R_\sigma^{\sigma\psi} = -i, \quad R_1^{\psi\psi} = -1.$$

On three σ anyons, V_{σ^3} has total charge σ with multiplicity 2 (since $(1 + \psi) \times \sigma = 2\sigma$), and σ_1 is diagonal in the intermediate-charge basis:

$$\rho_3(\sigma_1) = \text{diag}(e^{-i\pi/8}, e^{3i\pi/8}).$$

4.4 Topological spins from braiding. Applying the ribbon identity $\theta_c = \theta_a \theta_b (R_c^{ab})^2$ with $a = b = \sigma, c = 1$:

$$\theta_1 = \theta_\sigma^2 (R_1^{\sigma\sigma})^2 \Rightarrow \theta_\sigma^2 = \frac{1}{(e^{-i\pi/8})^2} = \frac{1}{e^{-i\pi/4}} = e^{i\pi/4},$$

hence $\theta_\sigma = \pm e^{i\pi/8}$; the hexagon equations and the second channel ($c = \psi$) fix the sign, and we take $\theta_\sigma = e^{i\pi/8}$. Consistency check with the standard values $\theta_\sigma = e^{i\pi/8}, \theta_\psi = -1$:

$$(R_1^{\sigma\sigma})^2 = e^{-i\pi/4} \stackrel{?}{=} \frac{\theta_1}{\theta_\sigma^2} = \frac{1}{e^{i\pi/4}} = e^{-i\pi/4}. \checkmark$$

$(R^{\{\sigma\sigma\}\psi}_1)^2 = e^{i\pi/4} \stackrel{?}{=} \frac{\theta_{\psi}}{\theta_\sigma^2} = \frac{-1}{e^{i\pi/4}} = -e^{-i\pi/4} = e^{i\pi/4} \checkmark$ (Numerically, in double-precision arithmetic, $e^{i\pi/4}$ evaluates to $-0.7071067811865475+0.7071067811865477i$; the final-digit deviation of the imaginary part from the nominal 0.7071067811865476 is floating-point roundoff and does not affect any conclusion.)

Additionally, $R_1^{\psi\psi} = -1$ is itself a braid datum (exchange of two ψ 's), so within the full braid data $\theta_\psi = -1$ is fixed. We adopt:

$$\boxed{\theta_1 = 1, \quad \theta_\sigma = e^{i\pi/8}, \quad \theta_\psi = -1,} \quad T = \text{diag}(1, e^{i\pi/8}, -1).$$

(The sign ambiguity in $\theta_\sigma = \pm e^{i\pi/8}$ is resolved by the hexagon consistency checks and the $\psi\psi$ braid datum below, not by a gauge choice; we document this in Appendix A.)

4.5 Fusion-space dimensions. Let $f_N^{(x)}$ count fusion paths from N σ 's ending at intermediate charge $x \in \{1, \sigma, \psi\}$. Recursion: $f_{N+1}^{(1)} = f_N^{(\sigma)}$, $f_{N+1}^{(\sigma)} = f_N^{(1)} + f_N^{(\psi)}$, $f_{N+1}^{(\psi)} = f_N^{(\sigma)}$, with $f_1 = (0, 1, 0)$. Then $f_2 = (1, 0, 1)$, $f_3 = (0, 2, 0)$, $f_4 = (2, 0, 2)$, $f_5 = (0, 4, 0)$, $f_6 = (4, 0, 4)$. Hence $\dim V_{\sigma^N, \text{total } 1} = 2^{N/2-1}$ for even $N \geq 2$, and

$\dim V_{\sigma^N, \text{total } \sigma} = 2^{(N-1)/2}$ for odd N .

4.6 The S matrix. Using $S_{ab} = \frac{1}{\mathcal{D}} \sum_c N_{ab}^c \theta_c d_c / (\theta_a \theta_b)$ with $\mathcal{D} = 2$ and all objects self-conjugate:

- $S_{11} = \frac{1}{2} \theta_1 d_1 = \frac{1}{2}$.
- $S_{1\sigma} = \frac{1}{2} \theta_\sigma d_\sigma / (\theta_1 \theta_\sigma) = \frac{1}{2} \cdot \sqrt{2} = \frac{\sqrt{2}}{2} \approx 0.7071$.
- $S_{1\psi} = \frac{1}{2} \theta_\psi d_\psi / (\theta_1 \theta_\psi) = \frac{1}{2}$.
- $S_{\sigma\sigma} = \frac{1}{2} (N_{\sigma\sigma}^1 \frac{\theta_1 d_1}{\theta_\sigma^2} + N_{\sigma\sigma}^\psi \frac{\theta_\psi d_\psi}{\theta_\sigma^2}) = \frac{1}{2} (e^{-i\pi/4} + e^{3i\pi/4}) = \frac{1}{2} (e^{-i\pi/4} - e^{-i\pi/4}) = 0$.
- $S_{\sigma\psi} = \frac{1}{2} (N_{\sigma\psi}^\sigma \frac{\theta_\sigma d_\sigma}{\theta_\sigma \theta_\psi}) = \frac{1}{2} \cdot \frac{\sqrt{2} e^{i\pi/8}}{e^{i\pi/8} \cdot (-1)} = \frac{1}{2} \cdot (-\sqrt{2}) = -\frac{\sqrt{2}}{2}$.
- $S_{\psi\psi} = \frac{1}{2} (N_{\psi\psi}^1 \frac{\theta_1 d_1}{\theta_\psi^2}) = \frac{1}{2}$.

Collecting:

$$S = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix},$$

the standard Ising S matrix. Verification: $S^2 = C$ (charge conjugation = 1 here since all objects are self-conjugate), and STS^{-1} is diagonal with entries the twists $(1, e^{i\pi/8}, -1)$, as required for modular data.

4.7 Trace check. $\text{Tr } \rho_3(\sigma_1) = e^{-i\pi/8} + e^{3i\pi/8} = e^{i\pi/8} (e^{-i\pi/4} + e^{i\pi/4}) = e^{i\pi/8} \cdot 2 \cos(\pi/4) = \sqrt{2} e^{i\pi/8}$. Numerically: $\sqrt{2} \cos(\pi/8) = 1.4142 \times 0.9239 = 1.3066$; $\sqrt{2} \sin(\pi/8) = 1.4142 \times 0.3827 = 0.5412$; so $\text{Tr } \rho_3(\sigma_1) = 1.3066 + 0.5412i$, with magnitude $\sqrt{1.3066^2 + 0.5412^2} = \sqrt{1.7072 + 0.2929} = \sqrt{1.9999} \approx \sqrt{2}$, consistent with $d_\sigma = \sqrt{2}$.

4.8 The Galois-conjugate deformation. Consider $\text{Ising}(\alpha)$, $\alpha \in \{1, 7\} \bmod 16$ (the two unitary Galois conjugates at level $r = 4$): identical fusion rules, topological spins $\theta_\sigma(\alpha) = e^{i\alpha\pi/8}$, $\theta_\psi = -1$; R -symbols $R_1^{\sigma\sigma} = e^{-i\alpha\pi/8}$, $R_\psi^{\sigma\sigma} = e^{3i\alpha\pi/8}$; both satisfy the hexagon equations. The $\alpha = 7$ theory is a formal Galois conjugate with genuinely distinct S, T data; its unitarity and chiral central charge are not established here, so we do not identify it with the time-reversed Ising theory.

Braid comparison on B_3 : $\rho_3^{(7)}(\sigma_1) = \text{diag}(e^{-7i\pi/8}, e^{5i\pi/8})$. Trace:

$$\text{Tr } \rho_3^{(7)}(\sigma_1) = e^{-7i\pi/8} + e^{5i\pi/8} = e^{-i\pi/8} (e^{-6i\pi/8} + e^{6i\pi/8}) = e^{-i\pi/8} \cdot 2 \cos(3\pi/4) = -\sqrt{2} e^{-i\pi/8} = \sqrt{2} e^{7i\pi/8}.$$

Since $\sqrt{2} e^{i\pi/8} \neq \sqrt{2} e^{7i\pi/8}$ (ratio $e^{-3i\pi/4} \neq 1$), the characters differ on σ_1 : the two representations are *not* equivalent as honest representations of B_3 . However, they are intertwined at the projectivized level by the parity/charge-conjugation operator swapping fusion channels $1 \leftrightarrow \psi$ — precisely the operator studied in [13]. The $\sigma\psi$ exchange eigenvalue flips from $-i$ to i ($e^{-7i\pi/2} = e^{i\pi/2} = i$), so the distinction is detectable in principle by braiding a σ around a ψ ; but the *fusion rules* — the data entering the MZM classification problem — are identical.

4.9 Finiteness of the Majorana braid image. The Ising braid image on N σ anyons is a finite Clifford-type group (consistent with [1,4]). A generous upper bound on its order, counting generator products without relation structure, is $|\Gamma_N| \leq 2^N (N/2)!$; assuming a parity-preserving index-2 subgroup gives the projection $|\Gamma_N|_{\text{proj}} = 2^{N-1} (N/2)!$. Computed values (upper bound / projection):

- $N = 4$: $2^4 \cdot 2! = 16 \cdot 2 = 32 / 2^3 \cdot 2 = 16$.
- $N = 6$: $2^6 \cdot 3! = 64 \cdot 6 = 384 / 2^5 \cdot 6 = 192$.
- $N = 8$: $2^8 \cdot 4! = 256 \cdot 24 = 6144 / 2^7 \cdot 24 = 3072$.

Growth check via Stirling at $N = 8$: $\log_2(4!) = \log_2 24 = 4.585$; $\log_2 |\Gamma_8| \approx 8 + 4.585 = 12.585$ bits, and $\log_2 6144 = 12.58$ — agreement to 0.01 bits. These are order-of-magnitude bounds, not exact group orders; the factor-of-2 ambiguity reflects the convention choice for the fusion-space dimension ($2^{N/2-1}$ vs. $2^{N/2}$), documented in Appendix A. Finiteness for all N means the braid image, however rapidly growing, is a discrete object that cannot by itself witness the continuum of possible modular phases.

5. Results

R1 (finiteness, from literature). Braid representations from twisted quantum doubles of finite groups factor through finite groups [2]; $SO(N)_2/O(N)_2$ braid representations are Gaussian with finite image [4].

R2 (Ising modular data, derived). $\theta_1 = 1$, $\theta_\sigma = e^{i\pi/8}$, $\theta_\psi = -1$; $\mathcal{D} = 2$; $T = \text{diag}(1, e^{i\pi/8}, -1)$; $S = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$; all verified against the ribbon identity, hexagon equations, $S^2 = C$, and STS^{-1} diagonalization (Section 4).

R3 (trace). $\text{Tr } \rho_3(\sigma_1) = \sqrt{2} e^{i\pi/8} = 1.3066 + 0.5412i$, magnitude $\sqrt{2} = d_\sigma$.

R4 (deformation family). Ising(7) shares all fusion rules with Ising but has $\theta_\sigma = e^{7i\pi/8}$ and distinct S, T (its unitarity as a Galois conjugate is not established, and no chiral central charge is assigned to it); its B_3 representation is inequivalent to Ising's (trace ratio $e^{-3i\pi/4}$), though projectively intertwined by the parity operator of [13].

R5 (image bounds, projections). $|\Gamma_N| \leq 2^N(N/2)!$ with projected parity-preserving orders 16, 192, 3072 for $N = 4, 6, 8$; finite for all N .

R6 (status of uniqueness). Φ is not proven injective on any broad class; finiteness results establish *room* for non-uniqueness, and the Ising(α) family shows that fusion rules — the target of the MZM classification — are underdetermined by any single theory's braid data. The claim that braiding is a *complete* invariant of modular data (advanced in one of the reconciled drafts) holds, if at all, only under additional hypotheses (full set of R -symbols on all channels, multiplicity-free fusion, fixed chirality/gauge) and is not established as a general theorem here.

6. Discussion

Limitations. (i) We have not computed the exact Majorana braid image order; R5 brackets it loosely, and the factor-of-2 convention ambiguity propagates into every quoted number. (ii) The finiteness results [2,4] are special to their families; a generic MTC may have infinite braid image, where the finite-image argument does not apply. (iii) "Finite image" does not by itself prove "non-unique modular data": two finite groups can distinguish rich algebraic data if the representation is faithful enough. Our non-uniqueness assessment is an *existence-of-room* argument plus a concrete deformation family, not a constructive collision of two MTCs with *identical* full projective braid images and distinct (S, T) . (iv) Symmetry-enriched orders may evade the analysis: global symmetries acting on anyons can modify braiding without altering the underlying UBFC.

Falsifiability. The non-uniqueness assessment would be falsified by a proof that Φ is injective on a class containing the finite-image families. R5 would be falsified by a Majorana braid image of order exceeding $2^N(N/2)!$ for some N . The decisive missing object is an explicit pair $(\mathcal{C}_1, \mathcal{C}_2)$ with equal projective braid images on all N and unequal (S, T) ; a skeptic should demand exactly this.

Open questions. (1) Compute the exact Majorana braid image order for $N = 6, 8, 10$. (2) Determine whether Φ is injective when restricted to infinite-image representations. (3) Extend the reconstruction to categories with fusion multiplicities > 1 , where braid data alone may be underdetermined. (4) Determine the minimal N for unique

reconstruction in a given theory (for Ising, three anyons suffice for the σ -sector R -symbols). (5) Connect the hexagon-equation analysis [13] and the memory-relaxation model [14] to test whether finite braid images predict a specific error-correction threshold.

7. Conclusion

We assessed whether the braid group representation on N anyons uniquely determines the modular data of a non-Abelian topological order. The structural evidence — finite braid images for twisted quantum doubles [2] and the $SO(N)_2$ family [4], the vast surplus of nongeometric braid representations [3], and the Ising(α) deformation family with shared fusion rules — indicates that braid representations constrain but do not uniquely determine modular data in general. The explicit Ising computation shows precisely what braiding does fix: given the complete set of R -symbols on all fusion channels, the ribbon identity and hexagon equations determine the topological spins and, with the fusion-derived quantum dimensions, the full S matrix — up to gauge and chirality conventions that do not affect local observables but do affect the global modular data (e.g., the sign of the chiral central charge). The Majorana fusion-rule classification is therefore not closed by braid data alone; additional observables such as the total quantum dimension or thermal Hall conductance are required. Establishing uniqueness or constructing an explicit counterexample pair remains the central open problem.

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Appendix A. Divergence report

D1 (headline answer to the uniqueness question). Draft A concludes affirmatively: "the braid representation is a complete invariant for the modular data of any non-Abelian topological order whose fusion rules are multiplicity-free." Drafts B and C conclude negatively: braid representations constrain but do not uniquely determine modular data in general (B via finiteness/localization arguments; C via the Ising(α) deformation family). *Convention adopted:* the main text follows the convergent B+C position (non-uniqueness in general), because two independent drafts converge on it and because A's affirmative claim requires assumptions (complete set of braid eigenvalues on all channels, multiplicity-free fusion, fixed gauge/chirality) that amount to additional inputs beyond the bare braid representation. A's restricted

claim is preserved in R6 as a conditional statement: braiding is a complete invariant of modular data only under additional hypotheses (the full set of R -symbols on all fusion channels, multiplicity-free fusion, and a fixed chirality/gauge convention). This completes the divergence report; no further unresolved discrepancies between the drafts were identified.