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# Braid Group Representations and the Uniqueness of Modular Data: The Ising Test Case and General Limitations

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## ABSTRACT

We investigate whether the unitary braid group representation carried by a collection of non-Abelian anyons uniquely determines the modular data—namely the  $S$  and  $T$  matrices—of the underlying modular tensor category (MTC). Using the Ising anyon theory as a concrete laboratory, we reconstruct the full modular data from the elementary braid eigenvalue  $R_{\sigma\sigma} = e^{-i\pi/8}$  together with the fusion rule  $\sigma \times \sigma = 1 + \psi$ . Explicit arithmetic yields the quantum dimension  $d_\sigma = \sqrt{2} \approx 1.4142$ , the total quantum dimension  $D = 2$ , the  $S$ -entry  $S_{\sigma 1} = \sqrt{2}/2 \approx 0.7071$ , the vanishing diagonal  $S_{\sigma\sigma} = 0$ , and the  $T$ -entry  $T_{\sigma\sigma} = e^{i\pi/8} \approx 0.9239 + 0.3827i$ . We also compute the fusion-space dimensions  $f(n, 1) = 2^{n/2-1}$  for even  $n$ , and the braid generator's ordinary order 16 (projective order 4). [...]

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## Abstract

We investigate whether the unitary braid group representation carried by a collection of non-Abelian anyons uniquely determines the modular data—namely the  $S$  and  $T$  matrices—of the underlying modular tensor category (MTC). Using the Ising anyon theory as a concrete laboratory, we reconstruct the full modular data from the elementary braid eigenvalue  $R_{\sigma\sigma} = e^{-i\pi/8}$  together with the fusion rule  $\sigma \times \sigma = 1 + \psi$ . Explicit arithmetic yields the quantum dimension  $d_\sigma = \sqrt{2} \approx 1.4142$ , the total quantum dimension  $D = 2$ , the  $S$ -entry  $S_{\sigma 1} = \sqrt{2}/2 \approx 0.7071$ , the vanishing diagonal  $S_{\sigma\sigma} = 0$ , and the  $T$ -entry  $T_{\sigma\sigma} = e^{i\pi/8} \approx 0.9239 + 0.3827i$ . We also compute the fusion-space dimensions  $f(n, 1) = 2^{n/2-1}$  for even  $n$ , and the braid generator's ordinary order 16 (projective order 4). By contrasting these positive results with families of braid representations that have finite image—twisted quantum doubles, Gaussian  $SO(N)_2$  representations, and infinite non-geometric embeddings—we formulate a **partial uniqueness theorem**: dense braid images determine  $(S, T)$  up to a global phase, whereas finite braid images leave essential ambiguities. The analysis clarifies the informational content of braiding experiments in Majorana-based platforms and delineates the need for complementary probes when braid images are non-dense.

## 1. Introduction

Topological phases of matter support anyonic excitations whose adiabatic exchange is described by a unitary representation of the braid group  $B_N$

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. An MTC encodes the full topological data in its modular  $S$  and  $T$  matrices, which together determine fusion rules, quantum dimensions, and braiding phases

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. A central question—explicitly raised in recent QNFO reports

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—is whether the braid representation on  $N$  anyons suffices to reconstruct  $S$  and  $T$  uniquely.

The practical relevance is immediate: experimental platforms based on Majorana zero modes (MZMs) implement projective braid operations

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; if these operations uniquely fix the modular data, braiding alone could certify the underlying topological order. Conversely, if distinct MTCs share identical braid images, additional probes (e.g., interferometry) would be required.

In this work we (i) formulate a concrete reconstruction pipeline for  $S$  and  $T$  from braid generators, (ii) apply it to the Ising anyon model, (iii) compute auxiliary quantities (fusion-space dimensions, braid order), and (iv) analyze broader families of braid representations from the literature to delineate when uniqueness holds. Section 2 surveys relevant prior results, Section 3 details the methodology, Section 4 presents explicit derivations, Section 5 reports the numerical outcomes, Section 6 discusses limitations and falsification conditions, and Section 7 concludes.

## 2. Background and Related Work

Unitary braid group representations derived from solutions of the Yang–Baxter equation have been studied extensively.

- **Localization of unitary braid group representations**

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connects unitary  $R$ -matrices to simple objects in unitary braided fusion categories, establishing locality constraints essential for topological quantum computation.

- **Twisted quantum doubles**

2

show that braid representations arising from finite groups always factor through finite groups, implying that the image of  $B_N$  can be too small to capture full modular data.

- **Gaussian  $SO(N)_2$  representations**

3

prove that braid representations on tensor powers of the spin object are Gaussian with finite image, limiting the information content of the braid group alone.

- **Infinite non-geometric braid embeddings**

4

construct an infinite family of braid representations via  $d$ -fold branched coverings, demonstrating that braid images can be highly non-universal.

- **Twisted tensor product constructions**

5

illustrate how gauging a pointed modular category can change modular data while preserving braid representations on the ungauged sector.

- **Defect-operator analyses in AdS/CFT**

6

connect braid group representations to holographic defect operators, emphasizing that braid data and modular data are related but not identical.

- **Spinning braid representations in the fractional quantum Hall effect**

7

generalize braid representations to particles with spin, showing that additional algebraic data (e.g., spin structures) lie beyond the braid group.

- **Generalized and quasi-localizations**

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emphasize the role of additional algebraic data (e.g.,  $F$ -symbols) beyond the braid group for reconstructing full modular data.

- **Classical meta-material realizations of non-Abelian defects**

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provide experimental platforms where braid representations are accessible but modular data may remain hidden.

- **Extended chiral  $su(2)$  WZNW model**

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offers a theoretical arena where fusion rings, braid operators, and modular data can be compared term by term.

- **Anyonic tensor-network simulations**

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enable numerical extraction of  $S$  and  $T$ , yet these methods rely on explicit knowledge of the underlying MTC.

Collectively, these works suggest that while braid representations are a necessary ingredient, they are not universally sufficient for reconstructing modular data. Our contribution is to make this statement precise for a representative set of models.

### 3. Methods

Our reconstruction proceeds in three steps:

1. **Extract braid eigenvalues.** For a simple object  $a$ , the elementary braid generator  $R_{aa}$  acts on the two-anyon Hilbert space  $V_{aa} = \bigoplus_c N_{aa}^c V_c$  with eigenvalues  $\theta_c/\theta_a$ , where  $\theta_x$  denotes the topological spin of  $x$

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1. **Determine quantum dimensions.** The quantum dimension  $d_a$  follows from the trace of the braid representation on  $V_{aa}$ :

$$\mathrm{Tr}(R_{aa}) = \sum_c N_{aa}^c \frac{\theta_c}{\theta_a} d_c.$$

Solving this linear system yields the total quantum dimension  $D = \sqrt{\sum_x d_x^2}$ .

1. **Construct  $S$  and  $T$ .** The  $T$  matrix is diagonal with entries  $T_{aa} = \theta_a$ . The  $S$  matrix obeys the Verlinde formula

$$N_{ab}^c = \sum_x \frac{S_{ax} S_{bx} S_{cx}^*}{S_{1x}},$$

which can be inverted to obtain  $S_{ax} = \frac{d_a d_x}{D} \frac{\theta_a \theta_x}{\theta_{a \otimes x}}$  for Abelian fusion channels. For non-Abelian channels we use the orthonormality condition

$$\sum_x S_{ax} S_{bx}^* = \delta_{ab}.$$

We apply this pipeline to the Ising MTC, whose braid eigenvalue  $R_{\sigma\sigma} = e^{-i\pi/8}$  is experimentally accessible in Majorana platforms

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. The fusion rules are  $\sigma \times \sigma = 1 + \psi$ , with 1 and  $\psi$  Abelian.

## 4. Analysis

### 4.1 Input data

| Symbol                                      | Meaning                                  | Value         | Source       |
|---------------------------------------------|------------------------------------------|---------------|--------------|
| $R_{\sigma\sigma}$                          | Braid eigenvalue for two $\sigma$ anyons | $e^{-i\pi/8}$ | 13           |
| $\theta_1$                                  | Topological spin of vacuum               | 1             | definition   |
| $\theta_\psi$                               | Topological spin of fermion $\psi$       | -1            | 13           |
| Fusion rule $\sigma \times \sigma$          | $1 + \psi$                               | —             | 13           |
| $d_1, d_\psi$                               | Quantum dimensions of 1 and $\psi$       | 1 each        | definition   |
| $d_\sigma$                                  | Quantum dimension of $\sigma$ (unknown)  | —             | to be solved |
| $N_{\sigma\sigma}^1, N_{\sigma\sigma}^\psi$ | Fusion multiplicities                    | 1 each        | 13           |

### 4.2 Determining $d_\sigma$

From the fusion rule,

$$d_\sigma^2 = \sum_c N_{\sigma\sigma}^c d_c = N_{\sigma\sigma}^1 d_1 + N_{\sigma\sigma}^\psi d_\psi = 1 + 1 = 2.$$

Hence

$$d_\sigma = \sqrt{2} \approx 1.4142.$$

### 4.3 Total quantum dimension $D$

$$D = \sqrt{d_1^2 + d_\sigma^2 + d_\psi^2} = \sqrt{1^2 + (\sqrt{2})^2 + 1^2} = \sqrt{1 + 2 + 1} = \sqrt{4} = 2.$$

### 4.4 Computing $T$ from monodromy

The braid eigenvalues on the two fusion channels are

$$r_1 = e^{-i\pi/8}, \quad r_\psi = e^{-i\pi/8} i = e^{3i\pi/8}.$$

Monodromy eigenvalues (double braid) are

$$m_1 = r_1^2 = e^{-i\pi/4}, \quad m_\psi = r_\psi^2 = e^{3i\pi/4}.$$

The ribbon relation  $m_c = \theta_c / \theta_\sigma^2$  gives

$$\theta_\sigma^2 = \frac{1}{m_1} = e^{i\pi/4} \Rightarrow \theta_\sigma = e^{i\pi/8},$$

choosing the principal root (the opposite sign corresponds to the global anomaly discussed in Section 6). The ratio  $m_\psi / m_1 = -1$  yields  $\theta_\psi = -1$ , consistent with a fermion. Thus

$$T = \text{diag}(1, e^{i\pi/8}, -1) = \text{diag}(1, 0.9239 + 0.3827i, -1).$$

### 4.5 Reconstructing $S$ from the Verlinde formula

Using  $D = 2$  and the known spins, we evaluate each entry.

- $S_{\sigma 1}$  (fusion channel  $c = \sigma$ ):

$$S_{\sigma 1} = \frac{d_\sigma d_1}{D} = \frac{\sqrt{2} \times 1}{2} = \frac{\sqrt{2}}{2} \approx 0.7071.$$

- $S_{\sigma\psi}$  (fusion channel  $c = \sigma$ ):

$$S_{\sigma\psi} = \frac{d_\sigma d_\psi}{D} \frac{\theta_\sigma}{\theta_\sigma \theta_\psi} = \frac{\sqrt{2} \times 1}{2} \frac{1}{-1} = -\frac{\sqrt{2}}{2} \approx -0.7071.$$

- $S_{\sigma\sigma}$  (channels  $c = 1, \psi$ ):

$$S_{\sigma\sigma} = \frac{1}{D} \left( \frac{\theta_1}{\theta_\sigma^2} d_1 + \frac{\theta_\psi}{\theta_\sigma^2} d_\psi \right) = \frac{1}{2} \left( e^{-i\pi/4} + \frac{-1}{e^{i\pi/4}} \right) = 0.$$

- $S_{11}, S_{\psi\psi}, S_{1\psi}$ : for Abelian objects the fusion channel is unique, so the Abelian formula  $S_{ax} = \frac{d_a d_x}{D}$  applies with no spin factor. Using the inputs of Section 4.1,  $d_1 = 1$ ,  $d_\psi = 1$ , and  $D = 2$  (Section 4.3), each of these entries is

$$S_{11} = \frac{d_1 d_1}{D} = \frac{1 \times 1}{2} = \frac{1}{2}, \quad S_{1\psi} = \frac{d_1 d_\psi}{D} = \frac{1 \times 1}{2} = \frac{1}{2}, \quad S_{\psi\psi} = \frac{d_\psi d_\psi}{D} = \frac{1 \times 1}{2} = \frac{1}{2},$$

In the same steps as for  $S_{\sigma 1}$  in Section 4.5, specialized to Abelian labels.

Arranging rows/columns in the order  $(1, \sigma, \psi)$ ,

$$S = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}.$$

All entries satisfy unitarity and the Verlinde eigenvalue relation.

#### 4.6 Fusion-space dimensions for multiple $\sigma$ anyons

Let  $f(n, 1)$  denote the dimension of the fusion space of  $n$   $\sigma$  anyons with total charge 1. Using the coupled recursion  $f(n, 1) = f(n-2, 1) + f(n-2, \psi)$  and  $f(n, \psi) = f(n-2, 1) + f(n-2, \psi)$ , with base values  $f(2, 1) = 1$  and  $f(2, \psi) = 1$  (since  $\sigma \times \sigma = 1 + \psi$ ), we obtain for even  $n$ :  $f(4, 1) = f(2, 1) + f(2, \psi) = 1 + 1 = 2$ ,  $f(6, 1) = f(4, 1) + f(4, \psi) = 2 + 2 = 4$ ,  $f(8, 1) = f(6, 1) + f(6, \psi) = 4 + 4 = 8$ , consistent with the closed form

$$f(n, 1) = 2^{n/2-1}, \quad n \text{ even.}$$

Thus  $f(2, 1) = 1$ ,  $f(4, 1) = 2$ ,  $f(6, 1) = 4$ ,  $f(8, 1) = 8$ , etc. For odd  $n$ ,  $f(n, 1) = 0$  and  $f(n, \sigma) = 2^{(n-1)/2}$ .

#### 4.7 Order of the elementary braid generator

The braid matrix on  $V_{\sigma\sigma}$  is

$$\rho_2(\sigma_1) = e^{-i\pi/8} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

The diagonal part satisfies  $\text{diag}(1, i)^4 = I$ , giving a **projective order 4**. The overall phase satisfies  $(e^{-i\pi/8})^{16} = 1$ , so the **ordinary order** is 16:

$$\rho_2(\sigma_1)^{16} = I_2, \quad \rho_2(\sigma_1)^8 = -I_2.$$

#### 4.8 Summary of derived quantities

| Quantity                       | Numerical value           | Interpretation                   |
|--------------------------------|---------------------------|----------------------------------|
| $d_\sigma$                     | $\sqrt{2} \approx 1.4142$ | Quantum dimension of $\sigma$    |
| $D$                            | 2                         | Total quantum dimension          |
| $S_{\sigma 1}$                 | 0.7071                    | Overlap of $\sigma$ with vacuum  |
| $S_{\sigma\psi}$               | -0.7071                   | Overlap of $\sigma$ with fermion |
| $S_{\sigma\sigma}$             | 0                         | Vanishing diagonal entry         |
| $T_{\sigma\sigma}$             | $0.9239 + 0.3827i$        | Topological spin of $\sigma$     |
| $f(n, 1)$ (even $n$ )          | $2^{n/2-1}$               | Fusion-space dimension           |
| $\text{ord}(\rho_2(\sigma_1))$ | 16 (projective 4)         | Braid generator order            |

These values match the canonical Ising modular data

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, confirming that the braid eigenvalue alone suffices to reconstruct  $S$  and  $T$  for this model.

## 5. Results

The explicit arithmetic in Section 4 yields the following concrete numerical results for the Ising anyon theory:

1. **Quantum dimension**  $d_\sigma = \sqrt{2} \approx 1.4142$ .
2. **Total quantum dimension**  $D = 2$ .
3. **Modular  $S$  matrix** (rows/columns ordered  $1, \sigma, \psi$ ):

$$S = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix},$$

giving  $S_{\sigma 1} = 0.7071$ ,  $S_{\sigma\psi} = -0.7071$ ,  $S_{\sigma\sigma} = 0$ .

#### 1. Modular $T$ matrix:

$$T = \text{diag}(1, e^{i\pi/8}, -1) = \text{diag}(1, 0.9239 + 0.3827i, -1).$$

1. **Fusion-space dimensions**  $f(n, 1) = 2^{n/2-1}$  for even  $n$ .

2. **Braid generator order** ordinary 16, projective 4.

All numbers arise directly from the braid eigenvalue  $R_{\sigma\sigma} = e^{-i\pi/8}$  and the fusion rule  $\sigma \times \sigma = 1 + \psi$ ; no additional simulations were performed.

## 6. Discussion

### 6.1 Scope of uniqueness

Our reconstruction demonstrates that for the Ising MTC the braid representation on two anyons uniquely determines the modular data up to an overall sign of  $\theta_\sigma$  (the **global anomaly**). This aligns with the intuition that **dense** braid images—those generating the full unitary group on the fusion space—encode enough information to invert the Verlinde relations.

However, the literature provides numerous counterexamples:

- **Twisted quantum doubles**

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yield braid representations that factor through finite groups, so distinct doubles can share identical braid matrices while possessing different  $S$  matrices.

- **Gaussian  $SO(N)_2$  representations**

3

have finite image; the spin-object braid representation is projectively identical across the whole family, yet the modular data varies with  $N$ .

- **Infinite non-geometric embeddings**

4

produce braid representations not arising from any MTC, showing that the class of braid representations exceeds that of topological orders.

- **Twisted tensor product constructions**

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illustrate that gauging a pointed category changes  $\mathcal{D}$  and  $S$  while leaving the ungauged braid sector unchanged.

These observations motivate the following **partial uniqueness theorem**:

**Theorem (Partial Uniqueness).**

Let  $\mathcal{C}$  be a unitary MTC and  $a \in \mathcal{C}$  a simple object. If the image of the braid representation  $\rho_n$  on the fusion space  $V_a^{\otimes n}$  is dense in  $U(\dim V_a^{\otimes n})$  for some  $n \geq 2$ , then the modular data  $(S, T)$  are uniquely determined (up to a global phase) by the braid eigenvalues and the fusion rules. If the image is finite, additional data (e.g.  $F$ -symbols or anomaly information) are required, and distinct MTCs may share the same braid representation.

## 6.2 Limitations and failure modes

- **Finite braid images.** When the image is finite, the eigenvalues provide only a discrete set of possibilities for the topological spins, leading to multiple admissible  $S$  matrices.
- **Gauge ambiguities.** The reconstruction assumes a fixed gauge for  $F$ -symbols; different gauge choices can alter intermediate phases while leaving physical  $S$  and  $T$  invariant. The sign ambiguity  $\theta_\sigma \rightarrow -\theta_\sigma$  is a concrete manifestation of this.
- **Experimental noise.** Realistic measurements of  $R_{aa}$  are subject to decoherence; small phase errors propagate non-linearly into  $S$  and  $T$ .
- **Higher-genus effects.** Our analysis is restricted to the planar braid group  $B_N$ . On surfaces of genus  $g > 0$ , additional mapping-class generators may be needed to fully constrain the modular data.
- **Non-unitary or non-semisimple theories.** The ribbon identities used here rely on unitarity; extensions to logarithmic CFTs remain open.

A falsifying experiment would identify two distinct MTCs that yield identical braid matrices for a generating set of anyons yet differ in their  $S$  matrices. Candidates include gauged vs. ungauged versions of a pointed category as in

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## 6.3 Open questions

1. **Classification of dense braid images.** Which families of MTCs admit dense braid representations? Preliminary evidence points to quantum-group categories at generic  $q$ .
2. **Robust reconstruction algorithms.** Can one devise a stable numerical inversion of the Verlinde formula tolerant to experimental uncertainties?
3. **Extension to non-unitary theories.** Our methods rely on unitarity; extending to logarithmic or non-semisimple categories is an open direction.
4. **Experimental resolution of the anomaly.** What minimal additional probe (e.g., interferometry) suffices to fix the sign of  $\theta_\sigma$  in practice?

## 7. Conclusion

Using the Ising theory as a fully explicit test case, we showed that the braid group representation—specifically the monodromy spectrum on the two-anyon fusion space, together with the fusion ring—reconstructs the complete modular data:  $S$  exactly, and  $T$  up to the single sign ambiguity  $\theta_\sigma \rightarrow -\theta_\sigma$ . The positive result is bounded by two structural facts from the literature: many braid representations have finite image

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, and gauging operations can relate distinct modular categories while preserving braid data

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. Consequently, braid representations constitute a nearly—but not fully—complete invariant of non-Abelian topological order; the residual ambiguity is precisely the gravitational anomaly invisible to braiding. For Majorana-based classification programs

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, the practical message is that braiding determines the fusion ring and  $S$  matrix but requires independent parity or anomaly input to fix  $T$ .

## References

[1] arXiv:1009.0241v2 — Localization of unitary braid group representations. [2] arXiv:math/0703274v1 — Braid group representations from twisted quantum doubles of finite groups. [3] arXiv:1401.5329v4 —  $SO(N)_2$  Braid group representations are Gaussian. [4] arXiv:2003.02496v1 — An infinite family of braid group representations. [5] arXiv:1906.08153v1 — Braid group representations from twisted tensor products of algebras. [6] arXiv:2505.16817v1 — Braid Group Representations and Defect Operators in AdS/CFT Correspondence. [7] arXiv:hep-th/9202024v1 — Spinning Braid Group Representation and the Fractional Quantum Hall Effect. [8] arXiv:1105.5048v1 — Generalized and quasi-localizations of braid group representations. [9] arXiv:1903.00463v2 — Topological braiding of non-Abelian mid-gap defects in classical meta-materials. [10] arXiv:0710.1063v3 — Zero modes' fusion ring and braid group representations for the extended chiral  $su(2)$  WZNW model. [11] arXiv:1708.06476v1 — Studies of braided non-Abelian anyons using anyonic tensor networks. [12] QNFO: Braid Group Representations, Modular Data, and the Classification. [13] QNFO: Majorana Zero Modes and Braiding Experiments. [14] QNFO: Reconciliation of Non-Uniqueness and Finite-Image Braid Representations.

## Appendix A. Divergence report

No divergent claims arose among the drafts; all quantitative statements are convergent. Single-appearance claims (e.g., fusion-space dimension formula, braid order) are retained as they are uniquely contributed by a single draft but do not conflict with any other source.

## Appendix B. Claim attribution

| Claim ID | Statement                                                                                                                            | Source drafts | Agreement  |
|----------|--------------------------------------------------------------------------------------------------------------------------------------|---------------|------------|
| C1       | $d_\sigma = \sqrt{2} \approx 1.4142$                                                                                                 | A, B, C       | CONVERGENT |
| C2       | Total quantum dimension $D = 2$                                                                                                      | A, B, C       | CONVERGENT |
| C3       | $S_{\sigma 1} = 0.7071$ (i.e. $\sqrt{2}/2$ )                                                                                         | A, B, C       | CONVERGENT |
| C4       | $T_{\sigma\sigma} = e^{i\pi/8} \approx 0.9239 + 0.3827i$                                                                             | A, B, C       | CONVERGENT |
| C5       | Fusion-space dimensions $f(n, 1) = 2^{n/2-1}$ for even $n$                                                                           | B             | SINGLE     |
| C6       | Braid generator $\rho_2(\sigma_1)$ has ordinary order 16 and projective order 4                                                      | B             | SINGLE     |
| C7       | $S_{\sigma\sigma} = 0$                                                                                                               | C             | SINGLE     |
| C8       | $S_{\sigma\psi} = -\sqrt{2}/2 \approx -0.7071$                                                                                       | C             | SINGLE     |
| C9       | Partial uniqueness theorem: dense braid image $\Rightarrow$ unique $(S, T)$ up to global phase; finite image $\Rightarrow$ ambiguity | A, B, C       | CONVERGENT |
| C10      | Non-uniqueness evidence from twisted quantum doubles, Gaussian $SO(N)_2$ , and gauging constructions                                 | A, B, C       | CONVERGENT |
| C11      | Sign ambiguity $\theta_\sigma \rightarrow -\theta_\sigma$ not fixed by braid alone                                                   | A, C          | CONVERGENT |