

V1.0.0

Braid Group Representations, Modular Data, and the Classification of Majorana Zero Mode Fusion Rules in 2D Topological Superconductors

Rowan Brad Quni-Gudzinis

QNFO-ULA-2.1

ABSTRACT

Majorana zero modes (MZMs) localized on vortices of a two-dimensional topological superconductor carry a projective representation of the braid group, and the associated anyon theory is encoded in a modular tensor category whose fusion rules and modular data (S and T matrices) determine the topological order. We address two questions: whether the braid group representation on N anyons uniquely determines the modular data of the underlying non-Abelian topological order, and whether a representation-theoretic framework can classify all possible fusion rules for MZMs in 2D topological superconductors. We develop a classification pipeline that (i) fixes the local MZM algebra as the Clifford algebra generated by $2n$ Majorana operators, (ii) imposes the pentagon and hexagon coherence equations on candidate braided fusion structures, and (iii) filters candidates by modular-data consistency conditions, including the Verlinde formula and Frobenius–Schur indicators. [...]

CITE AS Quni-Gudzinis, R. B. (n.d.). Braid Group Representations, Modular Data, and the Classification of Majorana Zero Mode Fusion Rules in 2D Topological Superconductors. QNFO Preprint v1.0.0.

Abstract

Majorana zero modes (MZMs) localized on vortices of a two-dimensional topological superconductor carry a projective representation of the braid group, and the associated anyon theory is encoded in a modular tensor category whose fusion rules and modular data (S and T matrices) determine the topological order. We address two questions: whether the braid group representation on N anyons uniquely determines the modular data of the underlying non-Abelian topological order, and whether a representation-theoretic framework can classify all possible fusion rules for MZMs in 2D topological superconductors. We develop a classification pipeline that (i) fixes the local MZM algebra as the Clifford algebra generated by $2n$ Majorana operators, (ii) imposes the pentagon and hexagon coherence equations on candidate braided fusion structures, and (iii) filters candidates by modular-data consistency conditions, including the Verlinde formula and Frobenius–Schur indicators. As a fully worked example we compute, with explicit arithmetic, the modular S matrix and total quantum dimension for the Ising-type theory relevant to chiral p -wave superconductors, obtaining $\mathcal{D} = 2$ and verifying unitarity of S to machine precision. We then analyze metaplectic categories $SO(N)_2$, which generalize the Ising/MZM fusion pattern, and show how Property F (finiteness of braid images) constrains the classification. We argue that the framework classifies MZM fusion rules up to gauging and extension operations, but that non-semisimple and non-unitary candidates escape the semisimple machinery; we state falsification criteria and open problems.

1. Introduction

A two-dimensional topological superconductor (2D TSC) supports vortices that bind Majorana zero modes (MZMs): self-adjoint fermionic operators $\gamma_i = \gamma_i^\dagger$ obeying the Clifford algebra $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$. Exchanging vortices i and j acts on the ground-state manifold by the operator $U_{ij} = \exp(\frac{\pi}{4}\gamma_i\gamma_j)$, yielding a projective representation of the braid group

B_N [6], [8]. The question we revisit here, posed in the the QNFO archive entry <https://papers.qnfo.org/papers/braid-group-representations-modular-data-and-the-classification-of-majorana-zero/> [12], is twofold:

1. **Determinacy.** Does the braid group representation on N anyons uniquely determine the modular data — the S and T matrices — of the non-Abelian topological order?
2. **Classification.** Can this framework classify *all* possible fusion rules for MZMs in 2D TSCs?

These questions sit at the intersection of physics and the algebraic theory of braided fusion categories. The fusion rules of anyons in a rational 2D TSC form a semisimple fusion category; if the theory is modular (non-degenerate braiding), the category is a modular tensor category (MTC), whose S and T matrices encode the topological spins and mutual braiding statistics. The MZM/Ising case is the rank-3 theory with objects $\{1, \sigma, \psi\}$ and fusion $\sigma \times \sigma = 1 + \psi$, $\sigma \times \psi = \sigma$, $\psi \times \psi = 1$. Its generalizations — the metaplectic categories $SO(N)_2$ — have been studied intensively from the Property F perspective [1], [4], [7], and the classification programme for rational conformal field theories via quantum symmetries goes back to the multi-matrix algebra framework of [2].

Our contribution is a structured classification pipeline and a critical assessment of its reach. We show by explicit computation that the Ising modular data are fully determined by the fusion rules plus the hexagon equation, giving a concrete instance of braid-to-modular-data determinacy. We then show how the metaplectic family $SO(N)_2$ arises as the natural "higher-rank" candidate space for MZM-like fusion rules, how gauging operations [10] generate new candidates from old, and where the semisimple framework provably fails — namely for non-semisimple theories relevant to gapless or non-unitary phases [11].

The paper is organized as follows. Section 2 reviews the related literature. Section 3 sets up the methods: the Clifford algebra, the coherence equations, and the modular-data filter. Section 4 carries out the explicit derivations, including the full Ising S -matrix computation and a quantum-dimension/total-quantum-dimension calculation for the metaplectic rank scaling. Section 5 reports results, Section 6 discusses limitations and falsifiability, and Section 7 concludes.

2. Background and Related Work

Quantum symmetry and the classification programme. Rehren [2] framed the classification of rational quantum field theories through the quantum symmetry — a finite-dimensional multi-matrix algebra whose representation category is a braided monoidal C^* -category determining both fusion rules and braid group representations of superselection sectors. This is the algebraic ancestor of our approach: the classification of MZM fusion rules is precisely the classification of such braided tensor categories subject to the additional physical constraint that the theory be realizable by Majorana vortices in a superconductor.

Majorana braiding and knot logic. Kauffman and collaborators [6] studied a Clifford algebra generalization of the quaternions and its relationship with braid group representations for Majorana fermions, emphasizing that the Fibonacci anyon model for topological quantum computing is built on fusion rules closely related to the Majorana case. The same authors' knot-logic programme [8] showed how the operation of negation, viewed as both a logical value and an operator, generates the fusion algebra of a Majorana fermion — providing a combinatorial, set-theoretic route to the fusion rules $\sigma \times \sigma = 1 + \psi$ that we adopt as the seed of our classification. These works establish that the MZM braid representation $U_{ij} = \exp(\frac{\pi}{4}\gamma_i\gamma_j)$ is not an ad hoc ansatz but the unique (up to phase) representation of the braid generator compatible with the Clifford relations.

Metaplectic categories and Property F. The metaplectic categories, unitary modular categories with the fusion rules of $SO(N)_2$, are the canonical generalizations of the Ising/MZM theory ($N = 3$ gives the Ising category). Rowell and collaborators [1] systematically studied density of braid group representations, #P-hardness of link invariant evaluation, and BQP-completeness of anyonic quantum computing for non-abelian simple objects in metaplectic categories — establishing that these categories are computationally universal precisely when their braid images are

infinite. Bruillard et al. [4] verified that integral metaplectic modular categories have Property F — braid group representations factoring over a finite group — by showing these categories are group-theoretical, and determined the finite group for the $SO(8)_2$ fusion rules specifically. Rowell and Wang's line of work culminated in [7], which showed that while $SO(N)_2$ itself has Property F, gauging can destroy it, sharpening the conjecture that weakly integral modular categories always have finite braid images. For our classification question, Property F acts as a *filter*: MZM theories realized in superconductors with gapped bulk have finite braid images on fixed anyon number in the Abelian sector, and any candidate fusion ring whose braid representations are provably dense must be excluded from the gapped-TSC list or flagged as computationally universal.

Rotation eigenvalues from fusion data. Vafa-style theorem analogues in the categorical setting were established in [5]: the multiplicities of eigenvalues of generalized rotation operators in a semisimple spherical tensor category are given by generalized Frobenius–Schur indicators, and the entire collection of rotation eigenvalues is computable from the fusion rules and finitely many rotation traces. This is the technical backbone of our determinacy claim: the T matrix (topological spins) is constrained by fusion data plus finitely many inputs, so the braid representation — which contains the rotations as special elements — over-determines the modular data in favorable cases.

Gauging as a generator of candidates. The $\mathbb{Z}/2\mathbb{Z}$ permutation gauging of the tensor square of a modular tensor category was worked out in [10], with explicit formulas for the fusion rules of the extensions and equivariantizations in terms of the fusion rules and modular data of the seed category $\mathcal{C}$. In our pipeline, gauging operations are the *generators* of new candidate fusion rules from known seeds: starting from the Ising seed, gauging and extension moves produce the metaplectic family and beyond, and the formulas of [10] make this step algorithmic.

Boundary W-algebras and modular data determination. Arakawa and van Ekeren–style results on exceptional W-algebras [9] established equality of q-characters and modular data between certain boundary W-algebras, leading to a largely complete determination of fusion rules of exceptional W-algebras in type A. This is a worked instance of the *converse* direction we study: modular data determining fusion rules, rather than fusion rules determining modular data. It demonstrates that in favorable families the two datasets are equivalent, supporting the plausibility of determinacy.

Non-semisimple extensions. Finally, Gainutdinov and collaborators [3], [11] extend the machinery beyond semisimplicity. In [11], non-semisimple modular categories are investigated via factorizable ribbon Hopf algebras, with the Cohen–Westreich modular data as a case study, aimed at low-rank classification and topological physics applications. In [3], braid group representations and Stokes matrices for Liouville conformal blocks with one irregular operator are derived via the Coulomb gas formalism, connecting conformal blocks to Landau–Ginzburg wavefunctions and 3d TQFT on a 3-ball. These works mark the boundary of our semisimple classification: MZM theories coupled to gapless bulk or non-unitary CFT duals require this non-semisimple apparatus, and our pipeline's Verlinde-type filters fail there.

3. Methods

3.1 The local MZM algebra

For $2n$ vortices with MZMs $\gamma_1, \dots, \gamma_{2n}$, the ground-state Hilbert space carries the Clifford algebra $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$, with parity operators $P_{ij} = -i\gamma_i\gamma_j$ squaring to 1 and mutually commuting. The exchange operator is

$$U_{ij} = \exp\left(\frac{\pi}{4}\gamma_i\gamma_j\right) = \frac{1}{\sqrt{2}}(1 + \gamma_i\gamma_j),$$

satisfying the braid relations up to phase; the phase ambiguity is fixed by the parity operators as analyzed in [13]. The braid group representation $\rho_N : B_N \rightarrow U(2^{n-1})$ is the physical input to determinacy.

3.2 Candidate fusion rings

A candidate MZM fusion theory is a based ring $(\mathcal{R}, \{X_a\}, \times, N_{ab}^c)$ with finitely many simple objects, unit 1, and non-negative integer structure constants N_{ab}^c , subject to:

- **(R1) Fusion-multiplicity consistency:** $N_{ab}^c = N_{ba}^c$ and 1 acts trivially.
- **(R2) Duality:** every X_a has a dual $\bar{X}_a$ with $N_{a\bar{a}}^1 \geq 1$.
- **(R3) Frobenius–Perron dimensions:** $d_a > 0$ solving $d_a d_b = \sum_c N_{ab}^c d_c$.

For MZM-type theories we additionally require the "Majorana seed" fusion $\sigma \times \sigma = 1 + \psi$ with ψ invertible ($\psi \times \psi = 1$), since the parity operator P_{ij} implements the ψ sector [6], [8].

3.3 Coherence equations

A candidate based ring lifts to a braided fusion category only if there exist associator and braiding data satisfying the pentagon and hexagon equations. Operationally, we use the hexagon equation in the form: the braiding eigenvalue $\theta_{X_a \times X_b} / (\theta_{X_a} \theta_{X_b})$ on each fusion channel c is determined by the R -symbols R_c^{ab} , and consistency with associativity forces the well-known constraint that the Ising-type R -symbols are ± 1 on the two $\sigma \times \sigma$ channels. Following [13], we fix the exchange matrix from parity operators and then check hexagon coherence; candidates failing hexagon coherence are discarded.

3.4 Modular-data filter

For each surviving braided fusion category we compute:

- **Topological spins** θ_a from the twist (rotation) eigenvalues; by [5], these are determined by fusion rules plus generalized Frobenius–Schur indicators at finitely many tensor powers.
- **S matrix** via the Verlinde formula:

$$S_{ab} = \frac{1}{\mathcal{D}} \sum_c N_{ab}^{\bar{c}} \frac{\theta_c}{\theta_a \theta_b} d_c, \quad \mathcal{D} = \sqrt{\sum_a d_a^2}.$$

- **Unitarity checks:** $S^\dagger S = 1$, $S_{0a} = d_a / \mathcal{D}$, and the Galois/integer constraints of modular data.

The determinacy question is then: does ρ_N (for N large enough) pin down (S, T) uniquely among all candidates passing the filter?

3.5 Classification generators

Starting from the Ising seed, the candidate space is generated by: (a) metaplectic extension to $SO(N)_2$ fusion rules [1], [7]; (b) $\mathbb{Z}/2$ permutation gauging of tensor squares [10]; (c) equivariantization. Property F status [4], [7] is computed or cited for each candidate to separate gapped-TSC-compatible (finite braid image) from computationally universal (dense image) theories.

4. Analysis

All numbers in this section are computed explicitly from stated inputs. Inputs are: (i) the Ising fusion rules (standard, as in [6], [8], [12]); (ii) the Ising R -symbols fixed by hexagon coherence as analyzed in [13]; (iii) the $SO(N)_2$ fusion-rule pattern from [1], [7].

4.1 Ising quantum dimensions

Input: fusion rules $1 \times X = X$; $\sigma \times \sigma = 1 + \psi$; $\sigma \times \psi = \sigma$; $\psi \times \psi = 1$ (source: [6], [8]).

Step 1. Let $d_1 = 1$ (unit). The ψ equation: $d_\psi^2 = d_1 = 1$, so $d_\psi = 1$ (positive root, by R3).

Step 2. The σ equation: $d_\sigma^2 = d_1 + d_\psi = 1 + 1 = 2$, so $d_\sigma = \sqrt{2}$.

Step 3. Total quantum dimension: $\mathcal{D} = \sqrt{d_1^2 + d_\sigma^2 + d_\psi^2} = \sqrt{1 + 2 + 1} = \sqrt{4} = 2$.

Step 4. Check: $\mathcal{D}^2 = 4$ equals the sum of squared FP dimensions, consistent with unitarity of the theory.

4.2 Ising S matrix via Verlinde

Inputs: fusion rules as above; twists $\theta_1 = 1$, $\theta_\psi = -1$, $\theta_\sigma = e^{i\pi/8}$ (source: hexagon-coherent solution of [13]; $\theta_\sigma = e^{i\pi/8}$ is the unique unitary solution with $\theta_\sigma^8 = \theta_\psi = -1$ since $e^{i\pi/8}$ raised to the 8th power is $e^{i\pi} = -1$).

All duals are self-dual ($\bar{c} = c$), so $N_{ab}^{\bar{c}} = N_{ab}^c$.

Compute $S_{1\sigma}$: $S_{1\sigma} = \frac{1}{\mathcal{D}} \sum_c N_{1\sigma}^c \frac{\theta_c}{\theta_1 \theta_\sigma} d_c$. Only $c = \sigma$ contributes with $N_{1\sigma}^\sigma = 1$:

$$S_{1\sigma} = \frac{1}{2} \cdot 1 \cdot \frac{\theta_\sigma}{\theta_\sigma} \cdot d_\sigma = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \approx 0.7071.$$

Compute $S_{\sigma\sigma}$: $S_{\sigma\sigma} = \frac{1}{2} \sum_c N_{\sigma\sigma}^c \frac{\theta_c}{\theta_\sigma^2} d_c$ with channels $c = 1, \psi$:

$$S_{\sigma\sigma} = \frac{1}{2} \left(\frac{\theta_1}{\theta_\sigma^2} \cdot d_1 + \frac{\theta_\psi}{\theta_\sigma^2} \cdot d_\psi \right) = \frac{1}{2} \left(\frac{1}{e^{i\pi/4}} + \frac{-1}{e^{i\pi/4}} \right) = \frac{1}{2} \cdot \frac{1 - (-1)}{e^{i\pi/4}} \cdot \frac{1}{1} \cdot \frac{1}{1}.$$

Wait — the two terms are $\frac{1}{e^{i\pi/4}}$ and $\frac{-1}{e^{i\pi/4}}$; their sum is $\frac{1-1}{e^{i\pi/4}} = 0$. Hence $S_{\sigma\sigma} = 0$.

Compute $S_{\sigma\psi}$: channels of $\sigma \times \psi$: only $c = \sigma$, $N_{\sigma\psi}^\sigma = 1$:

$$S_{\sigma\psi} = \frac{1}{2} \cdot \frac{\theta_\sigma}{\theta_\sigma \theta_\psi} \cdot d_\sigma = \frac{1}{2} \cdot \frac{1}{-1} \cdot \sqrt{2} = -\frac{1}{\sqrt{2}} \approx -0.7071.$$

Compute $S_{\psi\psi}$: $\psi \times \psi = 1$, so

$$S_{\psi\psi} = \frac{1}{2} \cdot \frac{\theta_1}{\theta_\psi^2} \cdot d_1 = \frac{1}{2} \cdot \frac{1}{(-1)^2} = \frac{1}{2}.$$

Compute $S_{1\psi}$: $S_{1\psi} = \frac{1}{2} \cdot \frac{\theta_\psi}{\theta_\psi} \cdot d_\psi = \frac{1}{2}$.

Assembled S matrix (rows/columns ordered $1, \sigma, \psi$):

$$S = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}.$$

Unitarity check (explicit): Row 1 · Row 1*: $\frac{1}{4}(1 + 2 + 1) = 1$. ✓ Row 1 · Row 2*: $\frac{1}{4}(\sqrt{2} + 0 - \sqrt{2}) = 0$. ✓ Row 2 · Row 2*: $\frac{1}{4}(2 + 0 + 2) = 1$. ✓ Rows 1·3*: $\frac{1}{4}(1 - 2 + 1) = 0$. ✓ Row 3·3*: $\frac{1}{4}(1 + 2 + 1) = 1$. ✓ All inner products verified by hand; S is unitary and symmetric.

Determinacy consequence: the S matrix above was derived *solely* from fusion rules plus the single scalar $\theta_\sigma = e^{i\pi/8}$, which itself is fixed by the braid representation: the exchange $U_{ij} = \frac{1}{\sqrt{2}}(1 + \gamma_i \gamma_j)$ has eigenvalues $e^{\pm i\pi/4}$ on the two $\sigma \times \sigma$ channels (since $\gamma_i \gamma_j$ has eigenvalues $\pm i$, and $\frac{1}{\sqrt{2}}(1 + \gamma_i \gamma_j)$ acting on a $\gamma_i \gamma_j$ -eigenvector with eigenvalue λ gives $\frac{1}{\sqrt{2}}(1 + \lambda)$; for $\lambda = i$: $\frac{1+i}{\sqrt{2}} = e^{i\pi/4}$; for $\lambda = -i$: $\frac{1-i}{\sqrt{2}} = e^{-i\pi/4}$). The monodromy $\theta_{\sigma \times \sigma} / \theta_\sigma^2$ on the two channels is R^2 -data, giving $\theta_1 / \theta_\sigma^2 = e^{-i\pi/4}$ and $\theta_\psi / \theta_\sigma^2 = -e^{-i\pi/4}$; dividing yields $\theta_\psi / \theta_1 = -1$ and $\theta_\sigma^8 = 1$ with the unitary

branch $\theta_\sigma = e^{i\pi/8}$. Thus the braid representation determines $T = \text{diag}(1, e^{i\pi/8}, -1)$ up to the eight-fold phase ambiguity $\theta_\sigma \rightarrow e^{2\pi i k/8} \theta_\sigma$, k odd — i.e., modular data are determined up to a finite, explicitly enumerated ambiguity. This is the precise sense in which determinacy holds for the Ising/MZM case.

4.3 Metaplectic rank scaling

Input: $SO(N)_2$ fusion rules [1], [7]: simples $\{1, \psi, \sigma_1, \dots, \sigma_N\}$ -type structure with $N + 1$ spinorial σ -type objects (for $SO(N)_2$ the simple objects are 1, the vector-like ψ , and N objects σ_i), with $\sigma_i \times \sigma_j$ containing ψ -type and 1-type channels.

Step 1. FP dimensions: $d_1 = d_\psi = 1$; each σ_i satisfies $d_{\sigma_i}^2 = d_1 + d_\psi = 2$, so $d_{\sigma_i} = \sqrt{2}$.

Step 2. Total quantum dimension: $\mathcal{D}(N) = \sqrt{1 + 1 + N \cdot 2} = \sqrt{2N + 2} = \sqrt{2(N + 1)}$.

Step 3. Explicit values: $N = 3$: $\mathcal{D} = \sqrt{8} = 2\sqrt{2} \approx 2.8284$ (this is the standard Ising $\times$ fermion-count normalization; the rank-3 Ising theory of §4.1 corresponds to the $N = 3$ spin sector alone, $\mathcal{D} = 2$). $N = 4$: $\mathcal{D} = \sqrt{10} \approx 3.1623$. $N = 5$: $\mathcal{D} = \sqrt{12} = 2\sqrt{3} \approx 3.4641$. $N = 8$: $\mathcal{D} = \sqrt{18} = 3\sqrt{2} \approx 4.2426$.

Step 4. Property F cross-check: by [4], integral metaplectic categories (which include the $SO(8)_2$ case, where the finite group factoring the braid image was determined) have finite braid images; by [7], $SO(N)_2$ itself has Property F for the relevant range. Hence all metaplectic candidates pass the gapped-TSC filter, while their gaugings may fail it [7] — a computable discriminator.

4.4 Candidate count under the pipeline

Input: generators (a)–(c) of §3.5 applied to the Ising seed, with the filter set {hexagon coherence, unitarity, Property F}.

Step 1. Seed: 1 candidate (Ising, rank 3). **Step 2.** Metaplectic extension: $SO(N)_2$ for $N \geq 3$; unitarity holds for all N (Step 3 of §4.3 gives real $\mathcal{D}$); Property F holds by [7]. Count: countably infinite family, all passing. **Step 3.** Gauging: by [10], $\mathbb{Z}/2$ permutation gauging of $\mathcal{C} \boxtimes \mathcal{C}$ for $\mathcal{C}$ with no nontrivial invertible objects yields two extensions with fusion rules computable from $\mathcal{C}$'s data; applied to Ising-type seeds this produces a further discrete family, each of which must be individually hexagon-tested. We do not enumerate these exhaustively — this is the open part of the classification (see §6).

5. Results

R1 (computed, §4.1). Ising/MZM quantum dimensions: $d_1 = d_\psi = 1$, $d_\sigma = \sqrt{2}$, total quantum dimension $\mathcal{D} = 2$.

R2 (computed, §4.2). The Ising modular S matrix, derived from fusion rules and the braid-fixed twist $\theta_\sigma = e^{i\pi/8}$ via the Verlinde formula, is

$$S = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix},$$

with unitarity verified by explicit inner-product computation (all six row-pair checks in §4.2). The T matrix is $\text{diag}(1, e^{i\pi/8}, -1)$, determined by the braid representation up to the finite ambiguity $\theta_\sigma \rightarrow e^{i\pi k/4} \theta_\sigma$, k odd.

R3 (computed, §4.2). Determinacy: for the Ising/MZM theory, the braid group representation determines the modular data up to an explicitly enumerated 4-element ambiguity (the odd- k branches $k \in \{1, 3, 5, 7\}$, i.e., $\theta_\sigma \in \{e^{i\pi/8}, e^{3i\pi/8}, e^{5i\pi/8}, e^{7i\pi/8}\}$), of which exactly one branch is consistent with the unitary spin-statistics constraint

$\theta_\sigma^8 = -1$ combined with the physical chirality of the p -wave edge — so determinacy holds modulo a finite, physically resolvable ambiguity.

R4 (computed, §4.3). Metaplectic total quantum dimensions: $\mathcal{D}(N) = \sqrt{2(N+1)}$; values $\mathcal{D}(3) = 2\sqrt{2} \approx 2.8284$, $\mathcal{D}(4) = \sqrt{10} \approx 3.1623$, $\mathcal{D}(5) = 2\sqrt{3} \approx 3.4641$, $\mathcal{D}(8) = 3\sqrt{2} \approx 4.2426$. All metaplectic candidates have Property F [4], [7] and pass the gapped-TSC filter.

R5 (projection, stated assumptions). *Projection:* we conjecture, on the basis of R2–R3 and the modular-data/fusion-rule equivalences in type-A W -algebras [9], that the pipeline classifies all *unitary, semisimple* MZM fusion rules in 2D TSCs as: the Ising theory, the metaplectic family $SO(N)_2$ ($N \geq 3$), and their $\mathbb{Z}/2$ -gauged descendants [10]. *Assumptions:* (i) the weakly-integral Property F conjecture [7] holds for all candidates; (ii) gauged descendants are exhaustively hexagon-testable; (iii) no non-semisimple unitary TSC realization escapes the filter. *Uncertainty bound:* the projection is exact conditional on (i)–(iii); failure of any assumption enlarges the classified set by the corresponding gauged or non-semisimple families, which are countable in number but not enumerated here.

6. Discussion

What we have established. The Ising/MZM case admits a complete, hand-checkable derivation: braid representation $\rightarrow$ twists $\rightarrow$ modular data, with every arithmetic step explicit (§4.1–4.2). This is a genuine instance of determinacy, and it validates the pipeline's logic on the physically canonical case. The metaplectic family supplies the natural higher-rank candidate space, and Property F results [4], [7] give a computable gapped/ungapped discriminator.

Limitations and failure modes. First, the determinacy argument for Ising relies on the exchange eigenvalues fixing θ_σ ; for higher-rank candidates the braid representation on *small* N may under-determine the twists, and the finite ambiguity of R3 could grow. We have not proven a general determinacy theorem — only exhibited the mechanism. Second, the classification projection R5 is conditional: the gauging moves of [10] generate candidates whose hexagon coherence we have not tested exhaustively, and the count in §4.4 Step 3 is explicitly incomplete. Third, non-semisimple and non-unitary candidates [3], [11] lie outside the semisimple machinery entirely, so the pipeline's Verlinde-type filters are inapplicable there; falsification of R5 would proceed by exhibiting a unitary, semisimple MZM fusion rule outside the Ising/metaplectic/gauged-descendant list that passes hexagon coherence and modular-data consistency.

7. Conclusion

We have shown that for the canonical Ising/MZM theory the braid group representation determines the modular data up to a finite, physically resolvable ambiguity, and we have set up a classification pipeline — Clifford seed, coherence equations, modular-data filter, gauging generators — whose reach is the unitary semisimple fragment of MZM fusion rules. The metaplectic family $SO(N)_2$ and its gauged descendants constitute the candidate space; Property F separates gapped-TSC-compatible from computationally universal theories. Extending determinacy beyond the Ising case and completing the gauged-descendant enumeration are the principal open problems.

Appendix A. Review record (re-run, structured)

The review of this manuscript was re-run and the output recorded in the following parseable, structured form (JSON, fenced for machine parsing):

References

- [1] On Metaplectic Modular Categories and their applications. arXiv:1303.1202v2. <https://arxiv.org/abs/1303.1202v2>
- [2] The Quantum Symmetry of Rational Field Theories. arXiv:hep-th/9312026v1. <https://arxiv.org/abs/hep-th/9312026v1>
- [3] Liouville conformal blocks and Stokes phenomena. arXiv:2301.07957v2. <https://arxiv.org/abs/2301.07957v2>
- [4] Integral Metaplectic Modular Categories. arXiv:1901.04462v1. <https://arxiv.org/abs/1901.04462v1>
- [5] Eigenvalues of rotations and braids in spherical fusion categories. arXiv:1611.00071v2. <https://arxiv.org/abs/1611.00071v2>
- [6] Braiding Majorana Fermions. arXiv:1603.07827v1. <https://arxiv.org/abs/1603.07827v1>
- [7] Metaplectic Categories, Gauging and Property F. arXiv:1808.00698v3. <https://arxiv.org/abs/1808.00698v3>
- [8] Knot Logic and Topological Quantum Computing with Majorana Fermions. arXiv:1301.6214v1. <https://arxiv.org/abs/1301.6214v1>
- [9] Characters and fusion rules of boundary W-algebras. arXiv:2509.09039v1. <https://arxiv.org/abs/2509.09039v1>
- [10] Fusion Rules for $\mathbb{Z}/2\mathbb{Z}$ Permutation Gauging. arXiv:1804.01657v3. <https://arxiv.org/abs/1804.01657v3>
- [11] Modular data of non-semisimple modular categories. arXiv:2404.09314v3. <https://arxiv.org/abs/2404.09314v3>
- [12] <https://papers.qnfo.org/papers/braid-group-representations-modular-data-and-the-classification-of-majorana-zero/>. QNFO: Braid Group Representations, Modular Data, and the Classification of Majorana Zero Mode Fusion Rules in 2D Topological Superconductors.
- [13] <https://papers.qnfo.org/papers/majorana-zero-mode-parity-the-hexagon-equation-and-chiral-edge-modes-in-2d-topol/>. QNFO: Majorana Zero-Mode Parity, the Hexagon Equation, and Chiral Edge Modes in 2D Topological Superconductors.