

Classical Floquet Drives from Quantized Cavity Fields: Error Budgets, Geometric Phases, Gauge Consistency, and Entanglement Bounds at Large Photon Number

Classical Floquet Drives from Quantized Cavity Fields: Error Budgets, Geometric Phases, Gauge Consistency, and Entanglement Bounds at Large Photon Number

Abstract

Floquet engineering treats periodically driven quantum systems with classical time-periodic Hamiltonians, while cavity quantum electrodynamics (QED) treats the drive itself as a quantized field. The precise connection between the two descriptions beyond weak coupling has remained subtle. Building on the program of [1], in which the cavity field is represented in a large-photon-number phase basis so that the time-dependent Schrödinger equation of a classically driven system emerges as a controlled limit of the fully quantized dynamics, we supply a quantitative bridge with three components. First, we derive the effective classical drive amplitude $\Omega = 2g\sqrt{N}$ and its finite-photon-number corrections, showing that the classical limit is approached as a power law in the mean photon number $\bar{n}$, not exponentially: the relative amplitude correction scales as $1/(2\sqrt{\bar{n}})$, bounded conservatively by $1/\sqrt{\bar{n}}$. Second, we quantify the gauge-inconsistency phase error per drive cycle incurred by ignoring photon-number quantization, obtaining $\pi/\sqrt{\bar{n}}$ to $2\pi/\sqrt{\bar{n}}$ radians, and connect the resulting geometric phase to the gerbe-level structure of adiabatic Floquet theory [4], with the mixed-state operational phase of [6] as the natural observable when fluctuations are retained. Third, we derive entanglement-depletion bounds tied to the Schmidt-gap phenomenology of pulsed Dicke dynamics [7]. For a representative cavity at $\bar{n} = 10^4$ we compute: an effective drive frequency of 10 ± 1.1 GHz for strong-coupling parameters ($g/\omega_c = 0.01$, $\omega_c/2\pi = 5$ GHz); a 0.5% relative amplitude correction; a per-cycle gauge phase error of 0.031–0.063 radians; and a classical-regime crossover at $\bar{n} \approx 3.9 \times 10^5$ photons for 0.01-radian phase resolution. These numbers convert the formal claims of [1] into falsifiable experimental targets.

1. Introduction

Periodic driving is a cornerstone of modern quantum control, enabling synthetic Hamiltonians, topological band structures, and dynamical stabilization. In the Floquet framework, a time-periodic Hamiltonian $H(t+T) = H(t)$ yields quasienergies that govern long-time dynamics. Cavity QED, by contrast, treats the field as a quantum degree of freedom with its own photon statistics and entanglement with matter. That the two descriptions agree at weak coupling is folklore; that they agree at the large photon numbers relevant to actual Floquet experiments is a quantitative question that has received surprisingly little systematic attention.

Reference [1] addresses exactly this gap: by expanding the cavity field in a large-photon-number phase basis, the authors show that the classically driven Schrödinger equation emerges directly from the fully quantized light–matter Hamiltonian, and they track the geometric-phase, gauge-consistency, and entanglement content that is invisible in the classical limit. The present paper is an independent quantitative companion to that program. Our aim is not to repeat the derivation but to make its error budget, gauge structure, and entanglement consequences explicit and numerical.

The central technical question we answer is: given a quantized cavity mode prepared in a coherent state with mean photon number $\bar{n}$, what is the leading-order discrepancy between the exact quantized dynamics and the classical-drive Floquet dynamics, expressed as an effective drive amplitude, an amplitude correction, an accumulated

geometric/gauge phase error per cycle, and a residual entanglement measure? We answer with fully shown arithmetic, distinguish derived results from labeled projections, and document all points where independent analyses of the same problem adopted different conventions (Appendix A).

2. Background and Related Work

The anchor reference. Reference [1] represents the cavity field in a large-photon-number phase basis and demonstrates that the time-dependent Schrödinger equation of a classically driven system emerges from the fully quantized light–matter Hamiltonian. Crucially, the construction is not merely a correspondence argument: it keeps track of the geometric phase acquired by the field sector and of the residual light–matter entanglement, thereby identifying precisely what the classical Floquet description discards. All quantitative work below builds on this structure.

Semiclassical validity and its breakdown. Reference [2] develops a quantum kinetic theory of light–matter interactions in degenerate plasmas and explicitly identifies the regime in which semiclassical models fail: large intensities and degenerate matter, where quantum-field fluctuations cannot be neglected even when intensities are large. This warns that "large photon number" does not automatically mean "classical": degeneracy and collective effects can amplify field fluctuations. Any claim that the Floquet limit is controlled by $\bar{n}$ alone must contend with this failure taxonomy; our error budget is the Floquet-case analogue of their regime boundary.

Geometric phases in Floquet theory. Reference [4] studies geometric phases in adiabatic Floquet theory using the (t, t') formalism, finding a double time integration that identifies the phases with horizontal lifts of surfaces in an abelian gerbe with connection, rather than with ordinary line-bundle holonomies. This is the closest structural precedent for the geometric phases of [1]: both works find that periodic driving promotes geometric phases from one-dimensional holonomies to two-dimensional surface objects. We use this in Section 3 to argue that the phase-basis representation inherits exactly this gerbe structure, with the photon-number phase playing the role of the extra fiber coordinate.

Mixed-state geometric phases. Reference [6] introduces an operational geometric phase for mixed quantum states based on spectrally weighted traces of holonomies, generalizing the standard interferometric definition. Since the classical limit of [1] involves tracing over or decohering the field, the field-reduced matter state is generically mixed, and the appropriate geometric phase for the emergent Floquet system is the operational mixed-state phase of [6] rather than the pure-state Berry phase. For a coherent field state, the spectral weights are Poissonian, $w_n = e^{-\bar{n}} \bar{n}^n / n!$, and the weighted phase average differs from the classical phase only at second order in the fluctuation fraction.

Entanglement in driven light–matter systems. Reference [7] studies the dynamics of entanglement and the Schmidt gap in the Dicke model under pulsed (time-dependent) light–matter coupling, showing that driving the coupling generates and probes quantum correlations inaccessible in static coupling, and can drive the system across quantum phase transitions in finite time. This supplies the observable — the Schmidt gap between the largest Schmidt coefficients of the light and matter partitions — with which the residual quantization of the drive can be measured: as $\bar{n}$ grows, the Schmidt-sector deviation from the classical trajectory should shrink in a computable way.

Gauge origins of interactions. Reference [8] derives the electron–phonon interaction by locally gauging the translational group in solids, so that a coupling usually postulated is instead generated by enforcing gauge invariance. This is philosophically aligned with [1]: in both cases the consistency conditions of a gauge principle are not optional constraints but the origin of the effective dynamics. Replacing a quantized cavity by a c-number drive is a gauge-fixing step, and the geometric phase records the redundancy that was fixed.

Community-level and experimental context. Reference [3], the Physics Briefing Book of the European Particle Physics Strategy Update, documents the community process by which large-scale priorities are set, evidencing that the strong-field light–matter frontier is strategically recognized ground for results like [1] and [2]. Reference [5], the Next Linear Collider report, establishes the experimental tradition of strong-field light–matter physics at e^+e^- colliders in the 500 GeV–1 TeV range, where high-photon-number regimes (beamstrahlung, laser-based acceleration) make the classical-vs-quantized field distinction practically relevant; we use its context in one labeled projection.

Programmatic context. References [9] and [10] develop thermodynamic frameworks for the viability and universality of quantum description levels and a thermodynamics of knowing; the classical limit of [1] is exactly the kind of level-transition whose cost these frameworks quantify. Reference [11] treats prime numbers as spectral artifacts, suggesting discrete spectral structures can carry unexpected arithmetic signatures — a speculative question for the phase basis of [1]. Reference [12] provides a unified fiber-bundle formalism for the Hopf fibration, the natural language for the coherent-state manifold, which is a complex projective space fibered over classical phase space. We note as a bibliographic limitation that [9–12] are engaged here at the level of titles and programmatic context.

3. Methods

3.1 Quantized model and phase-basis reduction

We adopt the standard model class of [1]: a two-level system (TLS) coupled to a single cavity mode,

$$H = (\hbar\omega_q/2)\sigma_z + \hbar\omega_c a^\dagger a + \hbar g(a + a^\dagger)\sigma_x, \quad (1)$$

with ω_q the qubit frequency, ω_c the cavity frequency, and g the coupling. Following [1], the cavity field is expanded in a phase (Pegg–Barnett-type) basis $|\phi_k\rangle$, $k = 0, \dots, N-1$, on a truncated photon space with $N \gg 1$, in which the annihilation operator acts approximately as multiplication by $e^{i\phi_k}$ with $\phi_k = 2\pi k/N$, so the drive term becomes $2\cos \phi \otimes \sigma_x$. Equivalently, for a coherent state $|\alpha\rangle$ with $\alpha = \sqrt{N} e^{i\phi}$, the field operators decompose as $\hat{a} \approx \sqrt{N} e^{i\phi} + \delta\hat{a}$ with fluctuations of order N^0 ; neglecting terms quadratic in $\delta\hat{a}$ yields the effective classical-drive Hamiltonian

$$H_{\text{eff}}(t) = (\hbar\omega_q/2)\sigma_z + \hbar\Omega \cos(\omega_c t + \phi_0) \sigma_x, \quad \text{with } \Omega = 2g\sqrt{N}. \quad (2)$$

Gauge consistency: the unitary $\hat{U}(t) = \exp[-i\phi(t)a^\dagger a]$ that shifts the cavity phase must leave physical observables invariant; under a $U(1)$ transformation of the matter states $\sigma_+ \rightarrow e^{i\chi}\sigma_+$, the phase state must transform covariantly for the reduced Hamiltonian to be gauge-invariant. The residual inconsistency is the commutator of the two transformations accumulated over one cycle, quantified in Section 4.

3.2 Fluctuation corrections

For a coherent state, $\text{Var}(n) = \bar{n}$ (Poissonian). The relative photon-number fluctuation is $\Delta n/\bar{n} = 1/\sqrt{\bar{n}}$. Differentiating the amplitude $|\alpha| = \sqrt{\bar{n}}$ gives $d|\alpha|/dn = 1/(2|\alpha|)$, so the relative drive-amplitude correction is $\delta|\alpha|/|\alpha| = 1/(2\sqrt{\bar{n}})$. The conservative bound used for quasienergy corrections is $|\delta\epsilon| \leq g \cdot \Delta n/\bar{n} = g/\sqrt{\bar{n}}$.

3.3 Geometric phase and gauge structure

For a cyclically driven TLS with drive phase winding once ($\phi: 0 \rightarrow 2\pi$), the dressed-state geometric phase in the adiabatic limit is $\gamma = \pi(1 - \cos \theta)$ with mixing angle $\tan \theta = 2g|\alpha|/(\hbar\Delta)$. In the quantized construction the field phase ϕ is itself a coordinate, and the total phase acquires a double-integral structure exactly as in the (t, t') theory of [4]:

one integration over the matter path in drive-parameter space, one over the field phase fiber. The consistent object is a U(1) gerbe connection with curvature two-form $F = dA \wedge d\phi$. The mixed-state observable is the operational phase of [6], reduced to its phase part, with Poissonian spectral weights.

3.4 Entanglement observables

Following [7], we use the Schmidt gap Δ_S of the matter–field bipartition. Two complementary quantifications are used: (i) a fluctuation-scale estimate of the Schmidt-gap deviation from separability, scaling as $2/\sqrt{\bar{n}}$ under a single-excitation leakage model; (ii) a depletion bound — the wavefunction weight discarded outside a classical band of half-width $k\sqrt{\bar{n}}$, bounded by Chebyshev $P \leq 1/k^2$ and, via the normal approximation to the Poisson distribution (valid at $\bar{n} = 10^4$ to relative accuracy $\sim 1\%$), by the Gaussian tail $P(|z| > 3) \approx 0.0027$.

All numerical inputs are stated assumptions, standard constants, or derived quantities; no simulated or measured data are used.

4. Analysis

All arithmetic is shown step by step.

Step 1: Cavity angular frequency (assumption: $\omega_c = 2\pi \times 5$ GHz, typical microwave-cavity scale). $\omega_c = 2\pi \times 5 \times 10^9 = 6.283185307 \times 5 \times 10^9 = 3.141592654 \times 10^{10}$ rad s⁻¹.

Step 2: Bare coupling (assumption: $g/\omega_c = 0.01$, strong-coupling circuit-QED scale). $g = 0.01 \times 3.141592654 \times 10^{10} = 3.141592654 \times 10^8$ rad s⁻¹.

Step 3: Photon-number amplitude (assumption: $\bar{n} = 10^4$, conservative laboratory-scale coherent drive). $\sqrt{\bar{n}} = \sqrt{(10^4)} = 100$.

Step 4: Effective drive amplitude (Eq. 2). $\Omega = 2g\sqrt{\bar{n}} = 2 \times 3.141592654 \times 10^8 \times 100 = 6.283185307 \times 10^{10}$ rad s⁻¹. Ordinary frequency: $\nu = \Omega/2\pi = 6.283185307 \times 10^{10} / 6.283185307 = 1.0 \times 10^{10}$ Hz = 10 GHz.

Step 5: Uncertainty propagation (assumption: 10% relative uncertainty on g and on $\bar{n}$). $\delta\Omega/\Omega = \sqrt{[(\delta g/g)^2 + (\frac{1}{2}\delta\bar{n}/\bar{n})^2]} = \sqrt{[0.10^2 + 0.05^2]} = \sqrt{0.0125} = 0.1118033988749895$. $\delta\nu = 0.1118033988749895 \times 10$ GHz = 1.118033988749895 GHz. Result: $\nu = 10 \pm 1.1$ GHz.

Step 6: Relative amplitude correction. $\delta|\alpha|/|\alpha| = 1/(2\sqrt{\bar{n}}) = 1/(2 \times 100) = 1/200 = 0.005$, i.e., 0.5%. Conservative bound (fluctuation fraction itself): $\Delta n/\bar{n} = 1/\sqrt{\bar{n}} = 0.01$, i.e., 1%.

Step 7: Gauge-inconsistency phase error per cycle. Coherent-state phase uncertainty: $\Delta\phi \approx 1/(2\sqrt{\bar{n}})$ (from $\Delta n \cdot \Delta\phi \gtrsim \frac{1}{2}$ with $\Delta n = \sqrt{\bar{n}}$). Over one cycle with winding number 1: $\delta\gamma_{\text{gauge}} = 2\pi \cdot \Delta\phi = \pi/\sqrt{\bar{n}} = \pi/100 = 0.0314$ radians (gauge condition enforced). Pessimistic bound (full fluctuation acting coherently, gauge condition unenforced): $\delta\gamma_{\text{gauge,max}} = 2\pi/\sqrt{\bar{n}} = 2\pi/100 = 0.0628$ radians. Range: 0.031–0.063 radians per cycle at $\bar{n} = 10^4$.

Step 8: Berry-phase sensitivity. At resonance ($\Delta = 0$), $\theta = \pi/2$, $\gamma = \pi$, and $d\theta/d|\alpha| = 0$, so the Berry phase is first-order insensitive to the Step-6 correction. Off resonance, take $\Delta = 2g|\alpha|/\hbar$ so $\tan \theta = 1$, $\theta = \pi/4$: $\gamma = \pi(1 - \cos \pi/4) = \pi(1 - 0.70711) = 0.9200$ rad. $d\theta/d \ln|\alpha| = \sin \theta \cos \theta = 0.5$, so $\delta\theta = 0.5 \times 0.005 = 0.0025$ rad, and $\delta\gamma = \pi \sin \theta \cdot \delta\theta = \pi \times 0.70711 \times 0.0025 = 0.00555$ rad per cycle. The Poissonian-weighted (operational, [6]) correction is second order: $\leq \pi \times (\Delta n/\bar{n})^2 = \pi \times 10^{-4} \approx 3.14 \times 10^{-4}$ rad for smooth phase-amplitude dependence.

Step 9: Quasienergy correction at weak-coupling parameters (assumption: $g/2\pi = 1$ MHz, illustrative circuit-QED Floquet scale; $\bar{n} = 10^4$; $\omega_c/2\pi = 5$ GHz). Absolute bound: $g \cdot \delta = (2\pi \times 10^6) \times 0.01 = 6.28 \times 10^4$ rad s⁻¹, i.e., 10 kHz. Relative to the drive frequency: $10^4/5 \times 10^9 = 2 \times 10^{-6}$ — negligible for spectroscopy; relative to g it is 1%, which matters for geometric-phase measurements.

Step 10: Entanglement observables. (i) Schmidt-gap deviation (single-excitation leakage model): $\Delta_S \approx 1 - 2/\sqrt{\bar{n}} = 1 - 0.02 = 0.98$, i.e., a 2% deviation from separability at $\bar{n} = 10^4$. (ii) Depletion bound: outside a 3σ classical band, Chebyshev gives $P \leq 1/9 \approx 0.111$ (loose); the normal approximation gives $P(|z| > 3) \approx 0.0027$. Worst-case linear accumulation over N_p periods: $\Delta(N_p) \leq 0.0027 \cdot N_p$, saturating the classical description after $\sim 1/0.0027 \approx 370$ periods (no resonance assumed).

Step 11: Crossover photon number (assumption: 0.01-radian interferometric phase resolution). Solve $2\pi/\sqrt{\bar{n}} = 0.01$: $\sqrt{\bar{n}} = 628.3$, $\bar{n} = 628.3^2 = 394,784 \approx 3.9 \times 10^5$. Below $\bar{n} \approx 4 \times 10^5$, the naive classical treatment's gauge error exceeds 0.01 radians and is in principle resolvable.

Step 12: Labeled projection to collider-scale photon densities (assumption: $\bar{n} \sim 10^{12}$ per mode, order-of-magnitude estimate for high-intensity laser/beamstrahlung regimes in the context of [5]; Gaussian statistics; no resonant enhancement). $\delta = 1/\sqrt{10^{12}} = 10^{-6}$; geometric-phase correction $\leq 3.14 \times 10^{-6}$ rad; classical regime safe for $\geq 10^6$ periods under the linear bound. Uncertainty: near parametric resonances the linear accumulation can be saturated, reducing the safe period count by up to two orders of magnitude.

5. Results

All numbers are computed in Section 4; projections are labeled.

1. **Effective drive (derived).** For $\omega c/2\pi = 5$ GHz, $g/\omega c = 0.01$, $\bar{n} = 10^4$: $\Omega = 2g\sqrt{\bar{n}}$ gives $\nu = 10 \pm 1.1$ GHz — a quantized field producing a classical-like drive at twice the cavity frequency, placing the system in the high-frequency driving regime where higher-order Floquet sidebands are expected, consistent with the geometric phase structure of [4].
1. **Convergence rate (derived).** The relative amplitude correction scales as $1/(2\sqrt{\bar{n}}) = 0.5\%$ at $\bar{n} = 10^4$ (conservative bound $1/\sqrt{\bar{n}} = 1\%$). Convergence to the classical Floquet limit is power-law in photon number, not exponential — the central quantitative claim of this paper.
1. **Gauge-inconsistency phase error (derived).** Per cycle: $\pi/\sqrt{\bar{n}} = 0.0314$ rad (gauge condition enforced) to $2\pi/\sqrt{\bar{n}} = 0.0628$ rad (unenforced) at $\bar{n} = 10^4$. These are the first explicit magnitudes attached to the gauge-consistency conditions of [1].
1. **Berry-phase error (derived).** Off resonance at $\theta = \pi/4$: 0.0056 rad per cycle, subdominant to the gauge error by a factor of ~ 6 –11; the operational ([6]) Poissonian-averaging correction is second order, $\leq 3.14 \times 10^{-4}$ rad.
1. **Quasienergy correction (derived).** At $g/2\pi = 1$ MHz, $\bar{n} = 10^4$: bound 10 kHz, i.e., 2×10^{-6} of the 5 GHz drive frequency but 1% of the coupling — negligible for spectroscopy, relevant for precision geometric-phase work.
1. **Entanglement (derived, two observables).** Schmidt-gap deviation $2/\sqrt{\bar{n}} = 2\%$ at $\bar{n} = 10^4$ ([7]-style observable); discarded weight outside a 3σ classical band ≈ 0.0027 per period, with worst-case linear accumulation saturating the classical description after ~ 370 periods.
1. **Crossover (derived).** The gauge error exceeds a 0.01-radian experimental resolution for all $\bar{n} < 3.9 \times 10^5$. Projection: at 0.001-radian resolution the resolvable regime extends to $\bar{n} < 3.9 \times 10^7$ (uncertainty dominated by the assumed resolution).
1. **Projection (labeled).** At $\bar{n} \sim 10^{12}$ (collider-scale context of [5]): all corrections $\leq 10^{-6}$; classical description safe for $\geq 10^6$ periods, with resonance effects the dominant uncertainty.
1. **Structural result (qualitative).** The quantized-drive geometric phase is a gerbe holonomy, not a line-bundle holonomy; the double-time structure of [4] is reproduced, and the correct mixed-state observable is the operational phase of [6].

6. Discussion

Validity of the classical mapping. Our derivation confirms that a quantized cavity mode populated with a large coherent photon number can be replaced by a classical periodic drive with amplitude $\Omega = 2g\sqrt{N}$, validating the heuristic used in experimental protocols that treat the cavity field as a classical control knob. But the mapping is parametrically controlled, not exact: the $1/\sqrt{\bar{n}}$ power-law corrections (0.5–1% in amplitude, 10^{-2} -radian-level in phase at $\bar{n} = 10^4$) are at or above current experimental sensitivity in strong-coupling cavities.

Breakdown of the large- N approximation. The expansion neglecting $\delta\hat{a}$ terms fails for photon numbers below $\bar{n} \sim 10^3$, where fluctuations become comparable to the mean field and the phase ceases to be well defined; the kinetic treatment of [2] predicts significant quantum corrections there, and its degenerate-matter amplification mechanisms could in principle invalidate the coherent-state treatment exactly in the strong-intensity regime, leaving no clean window between the "classical limit" and "semiclassical breakdown" regimes.

Gauge consistency and counter-rotating terms. The phase-removal unitary must preserve observables; inclusion of counter-rotating terms (neglected in the RWA) can introduce gauge-dependent Stark shifts, and if the matter coupling does not commute with H_m the fluctuation correction acquires additional counter-rotating contributions that we have bounded but not computed exactly. Failure to account for these would falsify the claim that the emergent drive is purely sinusoidal.

Entanglement vs. classicality. The effective-drive description discards TLS–field entanglement, yet pulsed coupling can generate non-trivial Schmidt gaps even when the drive appears classical [7]. Our two entanglement quantifications (2% Schmidt-gap deviation; 0.0027 per-period depletion) are complementary rather than competing: the former measures correlation structure, the latter bounds wavefunction weight outside the classical sector. The entanglement witness is the most accessible falsifiable diagnostic of the underlying quantum description.

Limitations. (i) Single-mode, single-qubit model; multimode or many-body extensions could change the scaling. (ii) The coherent-state assumption is ideal; excess phase noise would add to the $1/\sqrt{\bar{n}}$ gauge error, making our numbers lower bounds. (iii) The phase-uncertainty relation $\Delta\phi \approx 1/(2\sqrt{\bar{n}})$ is heuristic; a rigorous Pegg–Barnett treatment could shift constants by factors of order unity, not the scaling. (iv) The Chebyshev/normal tail estimates are crude; exact Poisson tails would likely tighten the 370-period figure substantially. (v) The gerbe identification is a structural argument from [4]; a full proof requires constructing the connection and curvature explicitly in phase-basis variables.

Failure modes and falsifiability. The framework fails if (a) the geometric phases are pure-gauge artifacts removable by a global phase-basis redefinition — then the gauge "error" is unobservable; (b) field decoherence faster than one cycle converts the coherent error into incoherent noise, consistent with the mixed-state analysis of [6]; (c) experiments at $\bar{n} \approx 10^4$ with phase resolution better than 0.03 radians find no cycle-accumulated discrepancy (falsifying practical significance), or observe drive-amplitude scaling deviating from $\sqrt{\bar{n}}$ (e.g., linear or saturating), or Schmidt-gap changes far exceeding the derived bounds without resonance.

Open questions. (1) Do the phases of [1] constitute gerbe holonomies requiring the higher-bundle language of [12]? (2) Can the operational mixed-state phase of [6] be applied directly to the field-traced Floquet system, and does it differ from the pure-state phase by more than the computed 0.0056 rad? (3) How do cavity loss and the kinetic depletion rates of [2] modify the effective Floquet Hamiltonian? (4) What is the description-level cost of the classical reduction within the thermodynamic-viability frameworks of [9,10]? (5) Do photon-number spectral structures carry arithmetic signatures as conjectured in [11]? (6) Can the large- N mapping extend to multimode cavities, where the gerbe becomes non-abelian in general?

7. Conclusion

We have converted the qualitative correspondence of [1] — classical Floquet dynamics emerging from quantized light–matter interaction in a large-photon-number phase basis — into a quantitative error budget. The effective drive amplitude is $\Omega = 2g\sqrt{N}$, yielding a 10 ± 1.1 GHz drive for strong-coupling parameters at $\bar{n} = 10^4$. The classical limit is approached as a power law: 0.5% relative amplitude correction, 0.031–0.063 radians per-cycle gauge phase error, 0.0056 radians Berry-phase error off resonance, and entanglement signatures at the 2% (Schmidt gap) and 0.0027 (depletion) level, with the classical description safe for ~ 370 periods in the worst case and the gauge error resolvable below $\bar{n} \approx 3.9 \times 10^5$. Structurally, the quantized-drive geometric phase lives on an abelian gerbe, with the operational mixed-state phase of [6] as the correct observable when fluctuations are retained. These numbers delineate, with explicit arithmetic, when classical Floquet engineering is quantitatively trustworthy and when quantized-field effects — geometric and entangling — must be retained, and they define a concrete, falsifiable research program at the intersection of Floquet engineering, cavity QED, and gauge-theoretic description levels.

References

[1] Floquet physics from quantized light-matter interaction: geometric phases, gauge consistency, and entanglement. arXiv:2609.21741v1. <https://arxiv.org/abs/2609.21741v1> [2] Quantum kinetic theory of light-matter interactions in degenerate plasmas. arXiv:2410.05917v2. <https://arxiv.org/abs/2410.05917v2> [3] Physics Briefing Book. arXiv:1910.11775v2. <https://arxiv.org/abs/1910.11775v2> [4] Geometric phases in adiabatic Floquet theory, abelian gerbes and Cheon's anholonomy. arXiv:0905.4584v2. <https://arxiv.org/abs/0905.4584v2> [5] Physics and Technology of the Next Linear Collider: A Report Submitted to Snowmass '96. arXiv:hep-ex/9605011v1. <https://arxiv.org/abs/hep-ex/9605011v1> [6] Operational geometric phase for mixed quantum states. arXiv:1302.1838v2. <https://arxiv.org/abs/1302.1838v2> [7] Dynamics of Entanglement and the Schmidt Gap in a Driven Light-Matter System. arXiv:1711.05182v1. <https://arxiv.org/abs/1711.05182v1> [8] The electron-phonon interaction from fundamental local gauge symmetries in solids. arXiv:1307.3571v1. <https://arxiv.org/abs/1307.3571v1> [9] DOI 10.5281/zenodo.18036068. QNFO: Thermodynamic Viability and the Universality of Feynman Matter. [10] DOI 10.5281/zenodo.18428950. QNFO: Thermodynamics of Knowing. [11] DOI 10.5281/zenodo.17566147. QNFO: Prime Numbers as Spectral Artifacts. [12] DOI 10.5281/zenodo.18387812. QNFO: UNIFIED FIBER BUNDLE FORMALISM FOR THE HOPF FIBRATION.

Appendix A. Divergence report

D1. Relative amplitude correction: 0.5% vs 1% at $\bar{n} = 10^4$.

- Draft B derives $\delta|\alpha|/|\alpha| = 1/(2\sqrt{\bar{n}}) = 0.005$ by differentiating $|\alpha| = \sqrt{\bar{n}}$ (the physically correct first-order amplitude response to a number fluctuation δn).
- Draft C uses the conservative bound $\delta = \Delta n/\bar{n} = 1/\sqrt{\bar{n}} = 0.01$, treating the full relative number fluctuation as an amplitude-fluctuation bound.
- Convention behind the disagreement: B computes the expected correction; C bounds the worst case. Resolution: main text reports both, with B's $1/(2\sqrt{\bar{n}})$ as the physical scaling and C's $1/\sqrt{\bar{n}}$ as the conservative bound (Steps 6, 9). No silent resolution.

D2. Gauge-inconsistency phase error per cycle.

- Draft B: $\delta\gamma = \pi/\sqrt{\bar{n}}$ (gauge condition enforced) to $2\pi/\sqrt{\bar{n}}$ (unenforced), i.e., 0.031–0.063 rad at $\bar{n} = 10^4$, from coherent-state phase uncertainty $\Delta\phi \approx 1/(2\sqrt{\bar{n}})$ times the winding number.

- Draft C: $|\langle \delta \gamma \rangle| \leq |\gamma^{\text{cl}}| \cdot \delta = \pi \times 0.01 \approx 0.0314$ rad, from the classical geometric phase scaled by the fluctuation fraction.
- Convention: B quantifies the phase-reference uncertainty of the field; C bounds the geometric-phase sensitivity to amplitude fluctuations. The two coincide numerically at the lower edge (0.031 rad) because $\pi/\sqrt{\bar{n}} = \pi \cdot (1/\sqrt{\bar{n}})$. Resolution: main text adopts B's range as the gauge-error statement and retains C's Berry-phase sensitivity (0.0056 rad off resonance; $\leq 3.14 \times 10^{-4}$ rad operational) as a distinct, complementary quantity.

D3. Entanglement signature: 2% Schmidt-gap deviation vs 0.0027 per-period depletion.

- Draft B: $\Delta_S \approx 1 - 2/\sqrt{\bar{n}}$, giving a 2% deviation from separability at $\bar{n} = 10^4$ (single-excitation leakage model).
- Draft C: discarded wavefunction weight outside a 3σ classical band ≈ 0.0027 per period (normal tail of the Poisson distribution), accumulating linearly to ~ 370 periods.
- Convention: B measures correlation structure (Schmidt gap); C measures probabilistic weight depletion. These are different observables, but both drafts present theirs as "the" entanglement signature of the classical limit. Resolution: main text presents both as complementary observables (Step 10, Result 6) and flags the model dependence of each (leakage model vs Gaussian tail approximation).

D4. Parameter sets for numerical evaluation.

- Draft A: $\omega_c/2\pi = 5$ GHz, $g/\omega_c = 0.01$, $\bar{n} = 10^4 \rightarrow$ effective drive $\nu = 10$ GHz.
- Draft C: $g/2\pi = 1$ MHz (illustrative circuit-QED Floquet scale), $\bar{n} = 10^4$, $\omega_c/2\pi = 5$ GHz $\rightarrow$ quasienergy bound 10 kHz.
- Convention: A evaluates the emergent drive amplitude in a strong-coupling cavity; C evaluates the correction budget at weak-coupling parameters, where the quasienergy correction is the relevant observable. Resolution: main text uses A's parameter set for the drive-amplitude, gauge-phase, and entanglement results (Steps 1–8, 10) and C's parameter set for the quasienergy correction (Step 9), reporting both explicitly. No silent resolution.