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Corrected Condensation Indices and the Sixteen-Fold Way: A Quantitative Bulk-Boundary Dictionary for Ising-Type Anyon Condensates

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ABSTRACT

A companion preprint (<https://papers.qnfo.org/papers/a-two-index-framework-for-the-bulk-boundary-correspondence-of-anyon-condensation/>) introduced computable indices for anyon condensation: the dimension ratio $\kappa = D_C^2/D_D^2$, the chirality-sensitive index $\mu = 2\Delta c$, and the non-Abelian fraction f_{NA} . We correct an arithmetic error in that work's Case III, the condensation of $\text{Ising} \times \overline{\text{Ising}}$: the parent total quantum dimension satisfies $D_C^2 = 16$, not 14, because the non-Abelian anyon $(\sigma, \bar{\sigma})$ has quantum dimension $d_{(\sigma, \bar{\sigma})} = \sqrt{2} \cdot \sqrt{2} = 2$, contributing $2^2 = 4$ to the sum. The condensing boson $(\psi, \bar{\psi})$ fixes $(\sigma, \bar{\sigma})$, which splits into two Abelian child anyons, so the child is the toric code with $D_D^2 = 4$, giving the normal condensate ratio $\kappa = 16/4 = 4$ and a child-side non-Abelian fraction $f_{\text{NA}} = 0$, not $1/3$. [...]

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Abstract

A companion preprint (<https://papers.qnfo.org/papers/a-two-index-framework-for-the-bulk-boundary-correspondence-of-anyon-condensation/>) introduced computable indices for anyon condensation: the dimension ratio $\kappa = D_C^2/D_D^2$, the chirality-sensitive index $\mu = 2\Delta c$, and the non-Abelian fraction f_{NA} . We correct an arithmetic error in that work's Case III, the condensation of $\text{Ising} \times \overline{\text{Ising}}$: the parent total quantum dimension satisfies $D_C^2 = 16$, not 14, because the non-Abelian anyon $(\sigma, \bar{\sigma})$ has quantum dimension $d_{(\sigma, \bar{\sigma})} = \sqrt{2} \cdot \sqrt{2} = 2$, contributing $2^2 = 4$ to the sum. The condensing boson $(\psi, \bar{\psi})$ fixes $(\sigma, \bar{\sigma})$, which splits into two Abelian child anyons, so the child is the toric code with $D_D^2 = 4$, giving the normal condensate ratio $\kappa = 16/4 = 4$ and a child-side non-Abelian fraction $f_{\text{NA}} = 0$, not $1/3$. We identify $\mu = 2\Delta c$ with Kitaev's sixteen-fold way label $\nu \bmod 16$, verify the identification at the anchor points $\nu = 0$ and $\nu = 1$, and show that κ and f_{NA} are mirror-blind while μ alone is signed. We replace the heuristic boundary fusion rule " $\psi \sim 1$ " with a module-category treatment over the condensate algebra, state a completeness conjecture for the index triple, and delimit what the indices cannot see: chirality enters only through μ , and nothing in $(\kappa, \mu, f_{\text{NA}})$ addresses operational control or readout, which for Ising anyons remain Clifford-only braiding and fusion-parity measurement.

1. Introduction

The bulk-boundary correspondence in $(2+1)$ -dimensional topological phases states that the topological data of a bulk determine the physics of its boundary. In the setting of anyon condensation — the process by which a subset of bulk anyons is forced to condense, producing a new topological order — this correspondence is usually stated structurally: the child category is a quotient or module category of the parent category [2]. The companion work [10] proposed to make the correspondence quantitative by attaching computable numbers to a condensation: the dimension ratio κ , the chirality index $\mu = 2\Delta c$ (where Δc is the change in chiral central charge across the condensation), and the non-Abelian fraction f_{NA} .

That work contained an arithmetic error in its Case III, the condensation of the product theory $\text{Ising} \times \overline{\text{Ising}}$, which is the minimal setting for studying boundaries carrying Majorana zero modes (the σ anyon of the Ising theory models a Majorana defect; experimental platforms pursuing Majorana physics are reviewed in [3], [4], [5]). The error propagated into the reported values $D_C^2 = 14$, $\kappa = 14/4$, and $f_{\text{NA}} = 1/3$. In this paper we correct the case: the correct parent dimension is $D_C^2 = 16$, the child is the toric code with $D_D^2 = 4$, the condensate is normal with $\kappa = 4$, and the child-side non-Abelian fraction is $f_{\text{NA}} = 0$. The correction is not merely cosmetic: it restores the interpretation of the condensate as a normal condensation and changes the classification statement that the indices support.

Beyond the correction, we make four contributions. First, we identify the chirality index $\mu = 2 \Delta c$ with Kitaev's sixteen-fold way label $\nu \bmod 16$ and verify the identification at directly computable anchor points. Second, we argue that the heuristic boundary fusion rule " $\psi \sim 1$ " used in [10] should be replaced by the module-category treatment of the condensate algebra. Third, we formulate a completeness conjecture — whenever $\mu > 0$ or $f_{\text{NA}} > 0$, boundary defect fusion is multi-channel — and prove a restricted version of it within the sixteen-fold way series. Fourth, we delimit what the indices do not measure: chirality enters only through μ , and nothing in $(\kappa, \mu, f_{\text{NA}})$ speaks to operational questions of control and readout.

2. Background and Related Work

Anyon condensation as a categorical bootstrap. Bhardwaj–Schembera-type treatments of anyon condensation [2] work abstractly: a modular tensor category $\mathcal{D}$ is obtained from a parent $\mathcal{C}$ by condensing a connected étale (condensable) algebra $A \subset \mathcal{C}$, and a bootstrap analysis derives the relation between $\mathcal{C}$ and $\mathcal{D}$ from physical requirements of locality, unitarity, and positivity of dimensions. Our corrected Case III is exactly an instance of this framework: the condensate algebra is $A = \mathbf{1} \oplus (\psi, \bar{\psi})$, and the child category is the category of local A -modules $\mathcal{C}_A^{\text{loc}}$. The splitting of the fixed-point anyon $(\sigma, \bar{\sigma})$ into Abelian child anyons is the standard fixed-point resolution required by the bootstrap, and we use this formalism to replace the heuristic " $\psi \sim 1$ " boundary rule of [10].

Bulk-boundary correspondence in SPT and response language. The effective-field-theory treatment of bosonic symmetry-protected topological phases in [1] derives purely topological bulk response actions whose boundary physics is fixed by bulk invariants, including chiral central charge. Our program is the anyon-condensation analogue: find bulk topological invariants (D_C^2 , Δc , fusion multiplicities) that quantitatively fix boundary data. The index $\mu = 2 \Delta c$ is the condensation analogue of the chiral response mismatch tracked in [1]: it is the only one of our three indices that is signed and therefore orientation-sensitive.

Anyon fundamentals and the toric code. The pedagogical treatment of Abelian and non-Abelian anyons in [8] supplies the definitions we use throughout — topological spin θ_a , quantum dimension d_a (the asymptotic growth rate of fusion spaces, equal to 1 exactly for Abelian anyons), total quantum dimension $D = \sqrt{\sum_a d_a^2}$, braid groups, and the toric code as the canonical exactly solvable model with Abelian excitations $\{1, e, m, \varepsilon\}$. The Ising anyon data $d_\sigma = \sqrt{2}$, $\sigma \times \sigma = 1 + \psi$, and the toric-code data $D^2 = 4$ are the numerical inputs to every computation in Section 4.

Condensation analogies beyond topological matter. Tachyon condensation in bosonic string theory [6] provides a structural analogy from a very different field: an unstable configuration condenses and the resulting vacuum has fewer effective degrees of freedom, with boundary-state formalism describing the process. We use this analogy only heuristically, to motivate the expectation that condensation reduces total quantum dimension by an integer ratio κ ; the corrected Case III with $\kappa = 4$ conforms.

Quantification methodology from adjacent fields. The companion framework [10] belongs to a broader program of replacing structural statements with computable multi-index summaries. The fusion of statistics and network science for functional brain networks [7] is an adjacent-field example where a single structural descriptor was replaced by a battery of complementary indices; we borrow the methodological lesson that distinct indices must be checked for

independence — here, chirality sensitivity. Similarly, the equivalence result between distance-based and RKHS-based statistics in [9] shows that apparently different summary statistics can coincide; our identification $\mu = \nu \bmod 16$ is an analogous statement, equating our index with an established classification label.

Majorana experimental context. The MAJORANA program [3], [4], [5] pursues neutrinoless double-beta decay with germanium detectors, targeting effective Majorana-neutrino mass sensitivity below 50 meV [4]. We cite this program not for its physics results but as the terminological anchor for "Majorana statistics" at boundaries: the boundary defects of the Ising theory carry Majorana zero modes, and the readout question we flag in Section 6 (fusion-channel parity measurement) is the topological-quantum-computing analogue of the parity measurements central to [3], [4], [5].

3. Methods

3.1 Definitions

Let $\mathcal{C}$ be a unitary modular tensor category (the parent) with simple objects a , quantum dimensions d_a , topological spins θ_a , and total quantum dimension

$$D_{\mathcal{C}}^2 = \sum_{a \in \mathcal{C}} d_a^2.$$

A condensation is specified by a connected étale algebra $A = \bigoplus_i n_i a_i$ in $\mathcal{C}$; the child category is $\mathcal{D} = \mathcal{C}_A^{\text{loc}}$, the category of local A -modules [2]. Following [10], the indices are

$$\kappa = \frac{D_{\mathcal{C}}^2}{D_{\mathcal{D}}^2}, \quad \mu = 2 \Delta c, \quad f_{\text{NA}} = \frac{N_{\text{NA}}^{\mathcal{D}}}{N_{\text{tot}}^{\mathcal{D}}},$$

where $\Delta c = c_{\mathcal{C}} - c_{\mathcal{D}}$ is the difference of chiral central charges between parent and child, $N_{\text{NA}}^{\mathcal{D}}$ counts non-Abelian simple objects of the child, and $N_{\text{tot}}^{\mathcal{D}}$ the total number of child simples. A condensate is called **normal** when κ is a positive integer.

3.2 The Ising theory and its conjugate

The Ising theory has simple objects $\{1, \sigma, \psi\}$ with

$$d_1 = 1, \quad d_{\sigma} = \sqrt{2}, \quad d_{\psi} = 1,$$

fusion $\sigma \times \sigma = 1 + \psi$, $\sigma \times \psi = \sigma$, $\psi \times \psi = 1$; spins $\theta_1 = 1$, $\theta_{\psi} = 1$ (so ψ is an Abelian boson), $\theta_{\sigma} = e^{i\pi/8}$; and chiral central charge $c_{\text{Ising}} = 1/2$. The conjugate theory $\overline{\text{Ising}}$ has objects $\{1, \bar{\sigma}, \bar{\psi}\}$ with the same dimensions, spins $\theta_{\bar{\sigma}} = e^{-i\pi/8}$, and $c_{\overline{\text{Ising}}} = -1/2$.

3.3 Computational procedure

For each case we: (i) list the parent simple objects and compute $D_{\mathcal{C}}^2$ by direct summation; (ii) identify the condensate algebra A and compute the orbits of the fusion action of A 's invertible components; (iii) resolve fixed-point orbits by splitting into simple child objects, checking consistency with $D_{\mathcal{D}}^2$ and with the bootstrap requirements of [2]; (iv) compute κ , μ , f_{NA} from the definitions. All arithmetic is shown explicitly in Section 4; no numerical simulation is used.

4. Analysis

Every input number below is stated with its source; every arithmetic step is shown.

4.1 Inputs

Input 1 (standard Ising data [8]): $d_1 = 1$, $d_\sigma = \sqrt{2}$, $d_\psi = 1$; $\sigma \times \sigma = 1 + \psi$; $c_{\text{Ising}} = 1/2$.

Input 2 (standard toric-code data [8]): anyons $\{1, e, m, \varepsilon\}$, all Abelian with $d_a = 1$; fusion $e \times m = \varepsilon$.

Input 3 (sixteen-fold way, Section 3.2 and [8]): odd- ν members have sectors $\{1, \psi, \sigma\}$ with $d_\sigma = \sqrt{2}$; even- ν members are Abelian; each unit of ν contributes $\Delta c = 1/2$, so $c = \nu/2 \pmod{8}$.

4.2 Parent total quantum dimension: $D_C^2 = 16$

The parent is $\mathcal{C} = \text{Ising} \times \overline{\text{Ising}}$ with $3 \times 3 = 9$ simple objects $(a, \bar{b})$, $a, b \in \{1, \sigma, \psi\}$. Quantum dimensions multiply across the product:

$$d_{(a, \bar{b})} = d_a \cdot d_{\bar{b}}.$$

The nine anyons and their dimensions are:

$$\begin{aligned} (1, 1) : 1, \quad (1, \bar{\sigma}) : \sqrt{2}, \quad (1, \bar{\psi}) : 1, \\ (\sigma, 1) : \sqrt{2}, \quad (\sigma, \bar{\sigma}) : 2, \quad (\sigma, \bar{\psi}) : \sqrt{2}, \\ (\psi, 1) : 1, \quad (\psi, \bar{\sigma}) : \sqrt{2}, \quad (\psi, \bar{\psi}) : 1, \end{aligned}$$

using in particular $d_{(\sigma, \bar{\sigma})} = \sqrt{2} \times \sqrt{2} = 2$. Then

$$D_C^2 = 1 + 2 + 1 + 2 + 4 + 2 + 1 + 2 + 1 = 16.$$

Equivalently, since total quantum dimension multiplies under products,

$$D_C^2 = D_{\text{Ising}}^2 \cdot D_{\overline{\text{Ising}}}^2 = (1^2 + (\sqrt{2})^2 + 1^2) \times (1^2 + (\sqrt{2})^2 + 1^2) = 4 \times 4 = 16.$$

The erroneous value $D_C^2 = 14$ in [10] arose from mis-evaluating $d_{(\sigma, \bar{\sigma})}$: the correct value is 2, whose square $2^2 = 4$ contributes 4 to the sum, not 2. The correct value is

$$\boxed{D_C^2 = 16.}$$

4.3 The condensate and the child category: $D_D^2 = 4$

The condensed algebra is $A = \mathbf{1} \oplus (\psi, \bar{\psi})$, where $(\psi, \bar{\psi})$ is an Abelian boson: $d_{(\psi, \bar{\psi})} = 1 \times 1 = 1$ and $\theta_{(\psi, \bar{\psi})} = \theta_\psi \theta_{\bar{\psi}} = 1 \cdot 1 = 1$. Since ψ has trivial mutual braiding with every Ising anyon, no anyon is confined; the child is the orbifold of $\mathcal{C}$ by the fusion action of $(\psi, \bar{\psi})$. The orbits are:

Orbits 3 and 4 have dimension $\sqrt{2}$, which cannot occur as a simple object of a consistent child; per the bootstrap of [2], the half-dimension orbits pair with the split components of the fixed-point orbit. The key computation is the square of the non-Abelian parent anyon, using $\sigma \times \sigma = 1 + \psi$ (Input 1) componentwise:

$$(\sigma, \bar{\sigma}) \times (\sigma, \bar{\sigma}) = (\sigma \times \sigma, \bar{\sigma} \times \bar{\sigma}) = (1 + \psi, 1 + \bar{\psi}),$$

which expands as

$$(1 + \psi) \otimes (1 + \bar{\psi}) = (1, 1) \oplus (1, \bar{\psi}) \oplus (\psi, 1) \oplus (\psi, \bar{\psi}).$$

All four summands are Abelian, each with $d = 1 \times 1 = 1$. The fixed-point anyon $(\sigma, \bar{\sigma})$ of dimension $d = 2$ therefore splits into **two** Abelian child simple objects of dimension 1 each (dimension bookkeeping: $2 = 1 + 1$), which together with the identifications of orbits 1 and 2 supply the four child anyons. The child is

$$\mathcal{D} = \{\mathbf{1}, e, m, \varepsilon\}, \quad d_{\mathbf{1}} = d_e = d_m = d_{\varepsilon} = 1,$$

with the toric-code fusion rules $e \times m = \varepsilon$ (Input 2); this is the unique assignment consistent with the condensate being a boson with trivial monodromy. Hence

$$D_D^2 = 1^2 + 1^2 + 1^2 + 1^2 = 4.$$

4.4 The indices for Case III

$$\kappa = \frac{D_C^2}{D_D^2} = \frac{16}{4} = 4,$$

a positive integer, so the condensate is **normal**. The child-side non-Abelian fraction is

$$f_{\text{NA}} = \frac{N_{\text{NA}}^{\mathcal{D}}}{N_{\text{tot}}^{\mathcal{D}}} = \frac{0}{4} = 0,$$

since all four child simples are Abelian. This corrects the prior claims of [10]; the value $1/3$ was an artifact of the erroneous $D_C^2 = 14$ combined with a miscount of child anyons. For reference, the **parent-side** non-Abelian weight fraction is

$$f_{\text{NA}}^{(\text{parent})} = \frac{(\sqrt{2})^2 + (\sqrt{2})^2 + 2^2}{16} = \frac{2 + 2 + 4}{16} = \frac{8}{16} = \frac{1}{2},$$

showing that all parent non-Abelian structure is absorbed by the condensate.

4.5 Chirality: μ for Case III

The parent central charge is $c_C = 1/2 + (-1/2) = 0$; the child (toric code) has $c_D = 0$. Hence

$$\Delta c = c_C - c_D = 0 - 0 = 0, \quad \mu = 2 \Delta c = 0.$$

Case III is chirality-neutral across the condensation, as it must be for a self-conjugate product parent condensing to a non-chiral child.

4.6 The sixteen-fold way identification

By Input 3, the ν -th member of Kitaev's sixteen-fold way has $c = \nu/2 \pmod{8}$. Taking $\Delta c = \nu/2$ for the representative with $0 \leq \nu < 16$,

$$\mu = 2 \Delta c = 2 \cdot \frac{\nu}{2} = \nu \pmod{16}.$$

Thus μ is exactly the sixteen-fold way label. We verify the two anchor points by direct computation:

- $\nu = 0$ (toric code): $c = 0$, so $\Delta c = 0$ and $\mu = 2 \cdot 0 = 0$. ✓
- $\nu = 1$ (Ising): $c = 1/2$, so $\Delta c = 1/2$ and $\mu = 2 \cdot \frac{1}{2} = 1$. ✓

For the Ising theory itself, the directly computed indices are

$$D_C^2 = 1^2 + (\sqrt{2})^2 + 1^2 = 1 + 2 + 1 = 4, \quad f_{\text{NA}}^{(\text{parent})} = \frac{(\sqrt{2})^2}{4} = \frac{2}{4} = \frac{1}{2}.$$

4.7 Mirror-blindness of κ and f_{NA}

Conjugating a theory ($\mathcal{C} \rightarrow \overline{\mathcal{C}}$) sends $\theta_a \rightarrow \theta_a^{-1}$ and $c \rightarrow -c$ but leaves all d_a invariant. Hence D_C^2 , κ , and f_{NA} are invariant under conjugation: they are **mirror-blind**. This is verified for Case III: replacing Ising by $\overline{\text{Ising}}$ and vice versa leaves $d_{(\sigma, \bar{\sigma})} = 2$ and $D_C^2 = 16$ unchanged, while flipping the sign of each factor's central charge. Only $\mu = 2\Delta c$ is signed and chirality-sensitive.

4.8 Restricted completeness test

Claim: within the sixteen-fold way series, $\mu \neq 0$ (equivalently ν odd) implies multi-channel bulk fusion structure.

Proof from stated inputs: if ν is odd, the theory contains σ with $\sigma \times \sigma = 1 + \psi$ (Input 3, reducing to Input 1 at $\nu = 1$); this fusion has two channels (1 and ψ), hence is multi-channel. If ν is even, the theory is Abelian (Input 3), all fusions are single-channel, and $f_{\text{NA}} = 0$ trivially since there are no non-Abelian anyons. ■

5. Results

R1 (Corrected Case III). For $\text{Ising} \times \overline{\text{Ising}} \rightarrow \text{toric code}$ (Sections 4.2–4.5):

$$D_C^2 = 16, \quad D_D^2 = 4, \quad \kappa = \frac{16}{4} = 4, \quad \mu = 0, \quad f_{\text{NA}} = 0.$$

The parent-side non-Abelian weight fraction is $f_{\text{NA}}^{(\text{parent})} = 1/2$ (Section 4.4).

R2 (Anchor points of the series). Computed directly:

theory	D_C^2	Δc	μ	f_{NA}
toric code ($\nu = 0$)	4	0	0	0
Ising ($\nu = 1$)	4	1/2	1	1/2 (parent-side)
Ising $\times \overline{\text{Ising}}$ (Case III)	16	0	0	0 (child-side)

R3 (Sixteen-fold way identification — labeled projection). Under the standard assumption $\Delta c = \nu/2 \pmod{1}$ for the sixteen-fold way representative with label ν (Input 3), we project $\mu = \nu \pmod{16}$ for all $\nu \in \{0, \dots, 15\}$. This projection is verified only at $\nu = 0$ and $\nu = 1$ (R2); the remaining values inherit the uncertainty of the central-charge formula, which is exact for the sixteen-fold way but whose translation into (κ, f_{NA}) requires case-by-case category data we have not summed here.

R4 (Restricted completeness). Within the sixteen-fold way series, the claim " $\mu \neq 0$ implies multi-channel fusion" is proved in Section 4.8 from Inputs 1 and 3. Outside this series the analogous statement is a conjecture supported only by the small sample of R1–R2; no counterexample was found among the cases examined.

6. Discussion

Limitations of the correction. The Case III correction rests on the splitting of the fixed-point anyon $(\sigma, \bar{\sigma})$ into two Abelian child anyons. This splitting is forced by consistency with the bootstrap of [2] and by the dimension bookkeeping $16 = 4 \times 4$, but the identification of the child as the toric code (rather than some other four-anyon Abelian theory) relies on the fusion rules of the split components, which we fixed by requiring $e \times m = \varepsilon$ with toric-code spins. A different assignment of spins to e, m would give a different Abelian child; we chose the toric code because it is the assignment consistent with the condensate being a boson with trivial monodromy.

What would falsify the claims. The identification $\mu = \nu \bmod 16$ (R3) would be falsified by any sixteen-fold way member whose chiral central charge deviates from $\nu/2 \bmod 8$ in a way that makes $2\Delta c \neq \nu \bmod 16$; more seriously, it would be falsified if the sixteen-fold way label were not determined by Δc alone (e.g., if two members with the same Δc carried different ν). The completeness conjecture (R4, outside the series) would be falsified by a single condensate with $\mu > 0$ or $f_{\text{NA}} > 0$ whose boundary defect fusion is single-channel; our evidence covers only the sixteen-fold way series and the three computed families, so the general claim should be treated as a conjecture. A concrete search strategy is: (1) enumerate all fermionic topological orders with $\nu \bmod 16 \neq 0$ from the sixteen-fold way; (2) construct all admissible condensate algebras using the formalism of [2]; (3) compute the resulting boundary defect fusion rules via the module-category approach.

Failure modes of the index framework. First, κ and f_{NA} are mirror-blind (Section 4.7): they cannot distinguish a theory from its conjugate, so any chirality-dependent boundary physics is invisible to them; only μ is signed, and it can vanish even when parent and child have distinct braiding structures. Second, computing f_{NA} requires knowledge of the full simple-object set of the child; where the child is not fully classified, the index cannot be evaluated. Third, our analysis presumes a single condensate algebra $A = \mathbf{1} \oplus (\psi, \bar{\psi})$; more intricate condensates with multiple generators could yield different κ and f_{NA} . Fourth, and importantly, the indices say nothing about **control and readout**: Ising anyon braiding generates only the Clifford group — a finite subgroup of unitary operations, insufficient for universal quantum computation by braiding alone — and readout proceeds by fusion-channel parity measurement, not by any quantity captured by $(\kappa, \mu, f_{\text{NA}})$. The experimental Majorana programs [3], [4], [5] make this concrete: their observable of interest (a parity-sensitive signal) has no analogue among our indices. A user of the framework should therefore not read $\kappa = 4$ or $f_{\text{NA}} = 0$ as statements about operational capability.

Against ourselves. One might argue that the Case III correction is a bookkeeping fix of limited conceptual import. We disagree in one respect and concede in another. We disagree because the corrected $\kappa = 4$ restores normality of the condensate, which is the property the classification table in [10] was meant to tabulate; the erroneous $\kappa = 14/4 = 3.5$ would have falsely flagged Case III as abnormal. We concede that $f_{\text{NA}} = 0$ versus $1/3$ changes no structural conclusion, since the child is Abelian either way. A second self-criticism: the replacement of the heuristic " $\psi \sim 1$ " boundary fusion rule by the module-category treatment ($\mathcal{C}_A^{\text{loc}}$) is here an argument from consistency, not a full derivation; a complete super-modular treatment of the boundary would be needed to make it rigorous, and we have not constructed one. Third, the analogy to tachyon condensation [6] and the methodological borrowings from [7] and [9] are heuristic; they motivate the framework but do not constrain it.

Open questions. (i) Does $\mu = \nu \bmod 16$ extend to time-reversal-twinned (super-modular) members of the series, where Δc is only defined modulo the mirror contribution? (ii) Is normality ($\kappa \in \mathbb{Z}_{>0}$) sufficient as well as necessary for a consistent bosonic condensate? (iii) Can a fourth, mirror-sensitive or readout-sensitive index be defined within the same categorical data, or is readout irreducibly outside the topological data? (iv) Is there a systematic classification of all condensate algebras for the sixteen-fold way, and how do the indices vary across that landscape?

7. Conclusion

We corrected the Case III computation of the two-index framework of [10]: the parent Ising $\times$ $\overline{\text{Ising}}$ has $D_C^2 = 16$ (not 14), the child toric code has $D_D^2 = 4$, and the resulting indices are $\kappa = 4$, $f_{\text{NA}} = 0$, and $\mu = 0$. The correction restores the normality of the condensate and removes the spurious non-Abelian fraction $1/3$. By embedding the condensation in the module-category formalism of [2], we replace the heuristic boundary fusion rule " $\psi \sim 1$ " with a mathematically grounded construction. We identified the chirality index $\mu = 2\Delta c$ with Kitaev's sixteen-fold way label $\nu \bmod 16$, verified at $\nu = 0$ and $\nu = 1$ and projected across the series, proved a restricted completeness statement within that series, and delimited what the indices cannot see: mirror asymmetry (invisible to κ and f_{NA}) and operational control

and readout (invisible to all three). These results sharpen the quantitative bulk-boundary map for anyon condensation and define concrete targets — mirror-sensitive diagnostics and exhaustive counterexample searches — for completing the classification program initiated in [10].

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