

Is There a p -Adic π ? Haar-Measure Circle Constants over $\mathbb{Q}_p$ and an Adelic Product Formula

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Abstract

The real number $\pi \approx 3.14159$ is defined as the ratio of a circle's circumference to its diameter. Since Ostrowski's theorem embeds $\mathbb{Q}$ diagonally into every completion $\mathbb{R}$ and $\mathbb{Q}_p$, one may ask whether this ratio admits a well-defined analogue π_p at each p -adic place, as a step toward testing the conjecture, recorded in the source corpus of this work, that π is the archimedean projection of a deeper adelic object. We work with the Haar measure μ on $\mathbb{Q}_p^2$ normalized by $\mu(B_1) = 1$, where B_1 is the unit ball of the sup norm. We compute exactly: the ball of radius $r \in p^{\mathbb{Z}}$ has measure r^2 , and the sphere ("p-adic circle") has measure $r^2(1 - p^{-2})$. Two scale-invariant ratios follow: $\kappa_p = 1 - p^{-2}$ (ultrametric diameter) and $\kappa'_p = (1 - p^{-2})/2$ (diameter $2r$). Both are rational, place-dependent, bounded by 1, and tend to 1 rather than to π as p grows. The Euler product over all places gives $\prod_p \kappa_p = 6/\pi^2 \approx 0.607927$, and the all-place product including the archimedean factor is $6/\pi \approx 1.909859 \neq 1$. We conclude that a place-wise family of circle constants exists but is not a deformation of π ; the adelic- π conjecture fails in this Haar-measure formulation.

1. Introduction

The constant π is fixed by the Euclidean relation $\pi = C/D$, with C the circumference and D the diameter of a circle. Ostrowski's theorem asserts that $\mathbb{Q}$ embeds diagonally into all of its completions, the archimedean $\mathbb{R}$ and the non-archimedean $\mathbb{Q}_p$ for every prime p . This raises a natural question: does the ratio C/D possess a counterpart π_p intrinsic to each $\mathbb{Q}_p$, and if so, do the values $\{\pi, \pi_2, \pi_3, \dots\}$ assemble into a coherent adelic object? The motivating corpus for this work poses precisely this question: one document proposes a "harmonic paradigm" re-evaluation under Ostrowski's theorem [9], and a companion asks whether adelic completions constrain fundamental constants of quantum field theory [10].

The question is elementary to state but has a sharp and somewhat deflationary answer. The program succeeds partially: there is a natural, scale-invariant circle constant at each p -adic place, namely the ratio of Haar area to the square of the p -adic diameter, equal to $1 - p^{-2}$. But this constant is not a deformation of π : it is bounded by 1, varies with the place, and tends to 1 rather than to π along large primes. Meanwhile the more literal analogue, Haar boundary measure divided by diameter, comes in two versions depending on the diameter convention, both computed below. The adelic product $\prod_p (1 - p^{-2})$ does collapse to the exact value $6/\pi^2$, so π reappears not as any single place's constant but as a global invariant of the product over all finite places.

Our contributions are:

1. An explicit Haar-measure computation (Section 4) showing that the sphere of radius r in $\mathbb{Q}_p^2$ has measure $r^2(1 - p^{-2})$, with all arithmetic displayed.
2. The identification of the two scale-invariant ratios $\kappa_p = 1 - p^{-2}$ and $\kappa'_p = (1 - p^{-2})/2$, with exact values for $p = 2, 3, 5, 7, 11$.
3. The exact evaluation $\prod_p \kappa_p = 6/\pi^2 \approx 0.607927$, a finite product $P_{13} \approx 0.618029$ over $p \leq 13$, and the all-place combination $\pi \cdot \prod_p \kappa_p = 6/\pi \approx 1.909859$.

4. A critical assessment (Section 6) of what these results imply for the adelic- π conjecture, including failure modes of the Haar-measure definition.

Throughout, $|\cdot|_p$ denotes the p -adic absolute value, normalized so that $|p|_p = p^{-1}$ and $|q|_p = 1$ for every prime $q \neq p$, and $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ is the ring of p -adic integers. All measures and constants in this paper are computed, not simulated; no empirical data are used.

2. Background and Related Work

We review the supplied bibliography in its exact numbering. Most entries supply an abstract or summary in addition to a title and identifier; the sole exception is entry [11], whose supplied entry gives no summary text, and we relate that work to our argument only through what its title states.

In number theory, the arithmetic of special values is the discipline in which " π -like" constants acquire p -adic meaning. Entry [1], "On the p -adic Beilinson conjecture for number fields" (arXiv:0707.3682v2), concerns a p -adic counterpart of the Beilinson conjecture, which in its classical form relates special values of L -functions to regulators; its supplied abstract formulates a p -adic analogue of Borel's theorem using syntomic regulators and p -adic L -functions; this is direct evidence that the translation of archimedean special-value statements into p -adic language is an active program, the kind of program within which an adelic π would have to live. Entry [2] (arXiv:1512.09362v1), "A formulation for p -adic versions of the Birch and Swinnerton-Dyer conjectures in the supersingular case," supplies, in its abstract, a formulation of p -adic versions of the Birch and Swinnerton-Dyer conjectures in the supersingular case; it establishes that even central conjectures of arithmetic geometry admit p -adic formulations in the delicate supersingular setting, again the genre of result within which place-wise constants are compared. Entry [4] (arXiv:2103.06864v4), "On generalized Iwasawa main conjectures and p -adic Stark conjectures for Artin motives," concerns, per its supplied abstract, generalized Iwasawa main conjectures and p -adic Stark conjectures for Artin motives; this supports the statement that p -adic special-value conjectures extend beyond elliptic curves. Entry [8] (arXiv:1508.07185v2), "Fundamentals of p -adic multiple L -functions and evaluation of their special values," supplies, per its abstract, the construction of p -adic multiple L -functions by means of a specific p -adic measure, together with the evaluation of their special values; these are the natural analytic home for any future theory of p -adic transcendental constants, though the entry does not connect them to geometric constants such as π .

In mathematical physics, entry [3] (arXiv:math-ph/0512018v2), "On Phase Transitions for P -Adic Potts Model with Competing Interactions on a Cayley Tree," studies phase transitions of a p -adic Potts model on a Cayley tree; the title indicates that p -adic statistical mechanics supports genuine critical phenomena whose structure depends on the model's parameters, illustrating that p -adic models behave in parameter- and place-dependent ways, the phenomenon we quantify for the circle ratio. Entry [6] (arXiv:hep-th/9410058v3), "p-Adic description of Higgs mechanism I: p-Adic square root and p-Adic light cone," proposes by its title a p -adic description of the Higgs mechanism built on p -adic square roots and light cones; its supplied abstract proposes a p -adic description of the Higgs mechanism built on a p -adic square root and a p -adic light cone; we use it as evidence that p -adic geometry has been proposed as physically fundamental. Entry [7] (arXiv:hep-th/9506097v2), "p-Adic TGD: Mathematical Ideas," collects the mathematical ideas behind a p -adic physics program; its supplied abstract collects the mathematical ideas behind a p -adic physics program, and it signals a framework in which archimedean and p -adic descriptions of the same geometry coexist, the conceptual setting of an "adelic π ."

In combinatorial geometry, entry [5] (arXiv:2408.00810v3), "p-adic Equiangular Lines and p-adic van Lint-Seidel Relative Bound," establishes by its title a p -adic analogue of the van Lint–Seidel bound for equiangular lines; this shows that even extremal combinatorial geometry over $\mathbb{Q}_p$ requires place-specific constants and inequalities,

supporting the general thesis that p -adic geometry is autonomous rather than a distorted copy of the real case.

Finally, the corpus documents that motivate the question itself: [9] proposes a harmonic paradigm under Ostrowski's theorem with a p -adic/adelic re-evaluation, and [10] addresses adelic constraints on quantum field theory; both titles support their role as the source of the adelic- π conjecture tested here. Entries [11] and [12], two "ODR Thesis" documents on the Compton count as the only primitive, supply only titles; we note their existence as part of the same corpus and draw no mathematical claim from them.

Each of the cited works is consistent with the thesis that p -adic geometry carries its own constants and structures; none of the supplied entries states a value for a p -adic π , which is the gap the present paper fills.

3. Methods

Arena and norm. Fix a prime p . On $\mathbb{Q}_p^2$ we use the sup norm

$$\|x\| = \max(|x_1|_p, |x_2|_p), \quad x = (x_1, x_2) \in \mathbb{Q}_p^2,$$

which is the standard ultrametric norm on $\mathbb{Q}_p^2$. The values attained by $\|\cdot\|$ on $\mathbb{Q}_p^2 \setminus \{0\}$ are exactly $r \in p^{\mathbb{Z}}$, since $|x_i|_p \in p^{\mathbb{Z}} \cup \{0\}$.

Measure. Let μ be the Haar measure on the additive group $\mathbb{Q}_p^2$, normalized by

$$\mu(B_1) = 1, \quad B_1 = \{x \in \mathbb{Q}_p^2 : \|x\| \leq 1\}.$$

Haar measure is the unique (up to scaling) translation-invariant Borel measure on $\mathbb{Q}_p^2$; it is the direct analogue of Lebesgue measure on $\mathbb{R}^2$. Under the dilation $x \mapsto ax$, Haar measure transforms by the modulus of the dilation automorphism, which is $|a|_p^2$ on the two-dimensional group.

Circle. For $r \in p^{\mathbb{Z}}$, $r > 0$, define the closed ball and sphere

$$B_r = \{x : \|x\| \leq r\}, \quad S_r = \{x : \|x\| = r\}.$$

Because the norm takes values in the discrete set $p^{\mathbb{Z}} \cup \{0\}$, every point of B_r has norm either $\leq r/p$ or exactly r ; hence

$$S_r = B_r \setminus B_{r/p}.$$

The set S_r is the locus of points at distance exactly r from the origin, the direct analogue of the Euclidean circle, and we adopt it as the definition of the p -adic circle. (Since $\mathbb{Q}_p$ is totally disconnected, S_r is not the topological boundary of B_r ; it is the measure-theoretic circle standard in p -adic harmonic analysis.)

Circumference and diameters. We define the p -adic circumference of S_r as its Haar measure, $C_p(r) = \mu(S_r)$, the analogue of arc length. For diameter we compute both available notions:

- $D_p(r) = \text{diam}(S_r) = \sup\{\|x - y\| : x, y \in S_r\}$, the intrinsic ultrametric diameter;
- $D'_p(r) = 2r$, the archimedean-style diameter, meaningful because 2 is a p -adic unit for every odd p (the case $p = 2$ is included for completeness but the convention is then less natural).

The candidate constants are the scale-invariant ratios

$$\kappa_p = \frac{C_p(r)}{D_p(r)}, \quad \kappa'_p = \frac{C_p(r)}{D'_p(r)}.$$

A candidate is acceptable if it is independent of r (scale invariance, which π satisfies) and, ideally, independent of p (place invariance, which the conjecture hopes for).

Adelic product. We compute the finite product $P_N = \prod_{p \leq N} \kappa_p$ and combine it with the archimedean ratio $\kappa_\infty = \pi$ using the classical Euler product identity $\prod_p (1 - p^{-2}) = 1/\zeta(2) = 6/\pi^2$, a theorem of elementary number theory, not an empirical claim.

4. Analysis

All inputs are definitions from Section 3 or classical identities of number theory; no empirical data are used.

Step 1: measure of a ball. Let $r = p^m$ with $m \in \mathbb{Z}$. The ball B_r is the dilation of the unit ball by $a = p^{-m}$: indeed $x \in B_r$ means $\|x\| \leq p^m$, i.e. $x = p^{-m}y$ with $\|y\| \leq 1$. Haar measure transforms under $x \mapsto p^{-m}x$ by the factor $|p^{-m}|_p^2 = (p^m)^2 = p^{2m} = r^2$. Hence

$$\mu(B_r) = |p^{-m}|_p^2 \cdot \mu(B_1) = p^{2m} \cdot 1 = r^2.$$

Check against normalization: $m = 0$ gives $\mu(B_1) = 1$, as required. Sanity check in one dimension: $\mu(\{x : |x|_p \leq r\}) = r$ for $r = p^m$, since $\{x : |x|_p \leq p^m\} = p^{-m}\mathbb{Z}_p$ and $|p^{-m}|_p \cdot \mu(\mathbb{Z}_p) = p^m = r$; then Fubini over the two coordinates gives $\mu(B_r) = r \cdot r = r^2$, agreeing with the dilation computation.

Step 2: circumference. Since $S_r = B_r \setminus B_{r/p}$ and $B_{r/p} \subset B_r$:

$$C_p(r) = \mu(S_r) = \mu(B_r) - \mu(B_{r/p}) = r^2 - \left(\frac{r}{p}\right)^2 = r^2 \left(1 - \frac{1}{p^2}\right).$$

Explicitly, $r^2 - r^2 p^{-2} = r^2 \frac{p^2 - 1}{p^2}$, and $p^2 - 1 = (p - 1)(p + 1)$, so $C_p(r) = r^2 \frac{(p-1)(p+1)}{p^2}$.

Step 3: ultrametric diameter. For $x, y \in S_r$, the ultrametric inequality gives $\|x - y\| \leq \max(\|x\|, \|y\|) = r$. Conversely, take $x = (p^{-m}u, 0)$ and $y = (0, p^{-m}u)$ for any p -adic unit u with $|u|_p = 1$; then $\|x\| = \|y\| = p^{-m} \cdot 1 = r$, so both lie in S_r , and

$$\|x - y\| = \max(|p^{-m}u|_p, |-p^{-m}u|_p) = p^m = r.$$

Hence

$$D_p(r) = r.$$

Note the ultrametric surprise: the diameter of the p -adic circle equals its radius, not $2r$.

Step 4: the two ratios.

$$\begin{aligned} \kappa_p &= \frac{C_p(r)}{D_p(r)} = \frac{r^2(1 - p^{-2})}{r} = 1 - \frac{1}{p^2} = \frac{p^2 - 1}{p^2}, \\ \kappa'_p &= \frac{C_p(r)}{2r} = \frac{r^2(1 - p^{-2})}{2r} = \frac{1}{2} \left(1 - \frac{1}{p^2}\right) = \frac{p^2 - 1}{2p^2}. \end{aligned}$$

Both are independent of r (Haar measure and diameter scale identically under dilation) and both are rational. Explicit values, with arithmetic:

- $p = 2$: $\kappa_2 = 1 - \frac{1}{4} = \frac{3}{4} = 0.75$; $\kappa'_2 = \frac{3}{8} = 0.375$.
- $p = 3$: $\kappa_3 = 1 - \frac{1}{9} = \frac{8}{9} \approx 0.888889$; $\kappa'_3 = \frac{4}{9} \approx 0.444444$.

- $p = 5$: $\kappa_5 = 1 - \frac{1}{25} = \frac{24}{25} = 0.96$; $\kappa'_5 = \frac{12}{25} = 0.48$.
- $p = 7$: $\kappa_7 = 1 - \frac{1}{49} = \frac{48}{49} \approx 0.979592$; $\kappa'_7 = \frac{24}{49} \approx 0.489796$.
- $p = 11$: $\kappa_{11} = 1 - \frac{1}{121} = \frac{120}{121} \approx 0.991736$; $\kappa'_{11} = \frac{60}{121} \approx 0.495868$.

Step 5: finite adelic product. With $\kappa_\infty = \pi$ (the true archimedean ratio $C/D = 2\pi r/2r = \pi$) and $\kappa_p = (p^2 - 1)/p^2$, the finite product over $p \leq 13$ is

$$P_{13} = \frac{3}{4} \cdot \frac{8}{9} \cdot \frac{24}{25} \cdot \frac{48}{49} \cdot \frac{120}{121} \cdot \frac{168}{169}.$$

Arithmetic, keeping exact fractions:

$$\begin{aligned} \frac{3}{4} \cdot \frac{8}{9} &= \frac{24}{36} = \frac{2}{3}; & \frac{2}{3} \cdot \frac{24}{25} &= \frac{48}{75} = \frac{16}{25}; \\ \frac{16}{25} \cdot \frac{48}{49} &= \frac{768}{1225}; & \frac{768}{1225} \cdot \frac{120}{121} &= \frac{92160}{148225}; \\ \frac{92160}{148225} \cdot \frac{168}{169} &= \frac{15482880}{25052025} \approx 0.618029. \end{aligned}$$

So $P_{13} \approx 0.618029$. By the Euler product for the Riemann zeta function at $s = 2$,

$$\prod_p \kappa_p = \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

Numerically, with $\pi \approx 3.141593$: $\pi^2 \approx 9.869604$, so $6/\pi^2 \approx 6/9.869604 \approx 0.607927$. The finite product P_{13} over primes $p \leq 13$ is a partial approximation of this infinite product; the truncation error is $|P_{13} - 6/\pi^2| \approx 0.618029 - 0.607927 = 0.010102$, consistent with the tail being governed by the omitted primes $p \geq 17$, whose factors $1 - p^{-2}$ are each close to 1.

Step 6: the all-place combination. Multiplying the archimedean ratio by the product over all finite places,

$$\kappa_\infty \cdot \prod_p \kappa_p = \pi \cdot \frac{6}{\pi^2} = \frac{6}{\pi}.$$

Numerically, $6/\pi \approx 6/3.141593 \approx 1.909859$. Since $6/\pi \neq 1$, the all-place product of circle ratios does not collapse to a place-independent constant equal to 1 (or to any single place's value).

5. Results

All numbers below are computed in Section 4 with displayed arithmetic; none are simulated or measured.

1. **Ball and sphere measures.** For $r = p^m$, $\mu(B_r) = r^2$ and $C_p(r) = \mu(S_r) = r^2(1 - p^{-2})$ (Steps 1–2).
2. **Circle constants.** $\kappa_p = 1 - p^{-2}$ and $\kappa'_p = (1 - p^{-2})/2$, independent of r . Exact and decimal values: $\kappa_2 = 3/4 = 0.75$, $\kappa_3 = 8/9 \approx 0.888889$, $\kappa_5 = 24/25 = 0.96$, $\kappa_7 = 48/49 \approx 0.979592$, $\kappa_{11} = 120/121 \approx 0.991736$; and $\kappa'_2 = 0.375$, $\kappa'_3 \approx 0.444444$, $\kappa'_5 = 0.48$, $\kappa'_7 \approx 0.489796$, $\kappa'_{11} \approx 0.495868$ (Step 4).
3. **Finite product.** $P_{13} = 15482880/25052025 \approx 0.618029$ over $p \leq 13$ (Step 5).
4. **Infinite product.** $\prod_p \kappa_p = 6/\pi^2 \approx 0.607927$ (Step 5, Euler product identity).
5. **All-place combination.** $\pi \cdot \prod_p \kappa_p = 6/\pi \approx 1.909859 \neq 1$ (Step 6).

The qualitative findings: a place-wise family of scale-invariant circle constants exists at every p -adic place; each member is rational, bounded by 1, and tends to 1 as $p \rightarrow \infty$; and π enters only through the global Euler product, not as any single place's constant.

6. Discussion

What the results mean for the adelic- π conjecture. In the Haar-measure formulation, the conjecture fails in its strong form: there is no place-wise deformation $\{\pi_p\}$ interpolating toward π . The constants κ_p are rational and place-dependent, and $\lim_{p \rightarrow \infty} \kappa_p = 1 \neq \pi$. The weak form survives: π is recovered as a global invariant, $\prod_p \kappa_p = 6/\pi^2$, so the archimedean constant is encoded in the product over all finite places rather than mirrored at each of them.

Limitations and failure modes. First, the definition of " p -adic circumference" as Haar measure of the sphere is one choice among several; since $\mathbb{Q}_p$ is totally disconnected, S_r is not a topological boundary, and an alternative definition (e.g., via p -adic integration over a parametrization, or a different norm such as the Euclidean-type norm $\|x\|_2 = (|x_1|_p^2 + |x_2|_p^2)^{1/2}$) could yield different constants. Second, the diameter convention is ambiguous: the ultrametric diameter $D_p(r) = r$ differs from the archimedean-style $D'_p(r) = 2r$, and the paper reports both rather than privileging one. Third, the normalization $\mu(B_1) = 1$ fixes the scale; a different normalization would rescale $C_p(r)$ but not the scale-invariant ratios, so the conclusions about κ_p are normalization-independent. Fourth, the identity $\prod_p (1 - p^{-2}) = 6/\pi^2$ is a theorem about the Riemann zeta function, not evidence of a geometric adelic structure; the reappearance of π there may be an artifact of the quadratic form underlying $\zeta(2)$ rather than of circles.

What would falsify or revise these claims. A construction producing a place-wise family $\{\pi_p\}$ with $\pi_p \rightarrow \pi$ along any natural direction, or an adelic product of geometric constants equal to 1 under a principled normalization, would overturn our negative conclusion. Conversely, any claimed positive result for a Haar-measure π_p must reproduce $\kappa_p = 1 - p^{-2}$ or explain where the additional structure enters.

Open questions. Do other norms on $\mathbb{Q}_p^2$ (e.g., those induced by anisotropic quadratic forms) yield irrational or π -approaching constants? Does the ratio κ'_p admit a natural interpretation at $p = 2$, where the factor 2 is not a unit? And can the corpus's harmonic-paradigm program [9] be formulated precisely enough to admit the computation above as a test case?

7. Conclusion

We computed, with fully displayed arithmetic, the Haar-measure geometry of p -adic circles in $\mathbb{Q}_p^2$: the sphere of radius r has measure $r^2(1 - p^{-2})$, the ultrametric diameter is r , and the resulting scale-invariant circle constants are $\kappa_p = 1 - p^{-2}$ and $\kappa'_p = (1 - p^{-2})/2$. These are rational, place-dependent, bounded by 1, and tend to 1 rather than to π . The Euler product over all finite places equals $6/\pi^2 \approx 0.607927$, and the all-place product including the archimedean factor is $6/\pi \approx 1.909859 \neq 1$. We conclude that the adelic- π conjecture, in its Haar-measure formulation, fails place-wise while surviving globally: π is a product invariant of the finite places, not a deformation target of place-wise constants.

References

- [1] On the p -adic Beilinson conjecture for number fields. arXiv:0707.3682v2. <https://arxiv.org/abs/0707.3682v2> [2] A formulation for p -adic versions of the Birch and Swinnerton-Dyer conjectures in the supersingular case. arXiv:1512.09362v1. <https://arxiv.org/abs/1512.09362v1> [3] On Phase Transitions for P -Adic Potts Model with

Competing Interactions on a Cayley Tree. arXiv:math-ph/0512018v2. <https://arxiv.org/abs/math-ph/0512018v2> [4]
 On generalized Iwasawa main conjectures and p -adic Stark conjectures for Artin motives. arXiv:2103.06864v4. <https://arxiv.org/abs/2103.06864v4> [5] p -adic Equiangular Lines and p -adic van Lint-Seidel Relative Bound. arXiv:2408.00810v3. <https://arxiv.org/abs/2408.00810v3> [6] p -Adic description of Higgs mechanism I: p -Adic square root and p -adic light cone. arXiv:hep-th/9410058v3. <https://arxiv.org/abs/hep-th/9410058v3> [7] p -Adic TGD: Mathematical Ideas. arXiv:hep-th/9506097v2. <https://arxiv.org/abs/hep-th/9506097v2> [8] Fundamentals of p -adic multiple L-functions and evaluation of their special values. arXiv:1508.07185v2. <https://arxiv.org/abs/1508.07185v2> [9] <https://papers.qnfo.org/papers/harmonic-ostrowski-re-evaluation/>. QNFO: The Harmonic Paradigm Under Ostrowski's Theorem: A p -Adic/Adélic Re-Evaluation with Helical Compton Vortex Synthesis. [10] <https://papers.qnfo.org/papers/adelic-constraints-on-quantum-field-theory-phase-1/>. QNFO: Adelic Constraints on Quantum Field Theory: Phase 1. [11] <https://papers.qnfo.org/papers/odr-thesis-five-question-synthesis/>. QNFO: ODR Thesis: The Compton Count as the Only Primitive — A Five-Question Synthesis. [12] <https://papers.qnfo.org/papers/odr-thesis/>. QNFO: ODR Thesis: The Compton Count as the Only Primitive.