

Majorana Zero-Mode Parity, the Hexagon Equation, and Chiral Edge Modes in 2D Topological Superconductors

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Abstract

In a two-dimensional topological superconductor hosting vortices with Majorana zero modes (MZMs), the exchange of vortices yields a projective representation of the braid group. We investigate whether the full anyonic exchange matrix is uniquely determined by the MZM parity operators and whether consistency with the hexagon equation of the Ising modular tensor category necessitates chiral edge modes. We show that the MZM algebra $\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij}$ fixes the non-Abelian structure of the exchange operator up to an overall Abelian phase. Completing the modular data and imposing the Ising hexagon equation requires a topological spin $\theta_\sigma = e^{i\pi/8}$ for the vortices, which via the Gauss-Milgram sum implies a chiral central charge $c = 1/2 \pmod{8}$. Non-chiral systems with $c = 0$ yield incompatible topological spins, demonstrating that the MZM parity operators alone do not uniquely fix the full anyonic exchange matrix. We discuss implications for extrinsic defects and anomaly inflow.

1. Introduction

Non-Abelian anyons in two-dimensional topological phases offer a route to topologically protected quantum computation. In a class of topological superconductors—exemplified by the chiral $p_x + ip_y$ state—vortices trap Majorana zero modes (MZMs) whose algebraic exchange realizes a projective representation of the braid group (Ivanov, 2001). The resulting anyon theory is the Ising theory, with anyon types $\{1, \sigma, \psi\}$, where σ labels the vortex and ψ labels the fermion.

A central question is whether the MZM parity operators—the only local algebraic data available from the zero-mode subspace—uniquely determine the full anyonic exchange matrix, including all phases. Moreover, the consistency of braiding with the hexagon equation, a defining axiom of modular tensor categories (MTCs), imposes constraints relating the topological spin to the chiral central charge. This raises the question of whether non-chiral superconductors can host genuine Ising anyons or whether chiral edge modes are mandatory.

In this work, we show that the MZM parity operators fix the non-Abelian part of the exchange matrix but leave an overall Abelian phase undetermined. Requiring the hexagon equation to hold with Ising fusion rules forces the topological spin $\theta_\sigma = e^{i\pi/8}$, which by the Gauss-Milgram sum implies $c = 1/2 \pmod{8}$. Non-chiral systems ($c = 0$) cannot satisfy this condition.

2. Background

2.1 Majorana zero modes and braid group representations

Consider $2N$ MZMs $\{\gamma_i\}_{i=1}^{2N}$ localized at vortices in a 2D topological superconductor. They satisfy the Clifford algebra

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij}, \quad \gamma_i^\dagger = \gamma_i.$$

The exchange of adjacent vortices i and $i + 1$ is implemented by the unitary (Ivanov, 2001)

$$U_{i,i+1} = \exp\left(\frac{\pi}{4} \gamma_i \gamma_{i+1}\right),$$

which acts as $\gamma_i \rightarrow \gamma_{i+1}$, $\gamma_{i+1} \rightarrow -\gamma_i$ on the zero-mode operators. The parity operator $P = i^N \prod_{k=1}^{2N} \gamma_k$ (with eigenvalues ± 1) labels the two degenerate ground states in the zero-mode subspace.

2.2 Ising anyon theory

The Ising anyon theory has fusion rules

$$\sigma \times \sigma = 1 + \psi, \quad \sigma \times \psi = \sigma, \quad \psi \times \psi = 1.$$

The F -symbols are determined up to gauge freedom, and the R -symbols $R_1^{\sigma\sigma}$ and $R_\psi^{\sigma\sigma}$ encode the exchange phases in the two fusion channels. The topological spin θ_a of anyon a satisfies $\theta_1 = 1$, $\theta_\psi = -1$, and θ_σ is a priori constrained by consistency. For the standard Ising MTC, $\theta_\sigma = e^{i\pi/8}$, giving $R_1^{\sigma\sigma} = e^{-i\pi/8}$ and $R_\psi^{\sigma\sigma} = e^{3i\pi/8}$ (Kitaev, 2006).

2.3 Hexagon equations

The hexagon equations are consistency conditions relating the F - and R -symbols of a braided tensor category. For three σ anyons, the hexagon equation in the fusion tree basis reads (Kitaev, 2006):

$$R_e^{ac} R_d^{bc} F_d^{abc} \begin{bmatrix} e \\ \cdot \end{bmatrix} = \sum_f F_d^{abc} \begin{bmatrix} \cdot \\ f \end{bmatrix} R_f^{ab} F_d^{abc} \begin{bmatrix} f \\ \cdot \end{bmatrix},$$

where the sum runs over fusion outcomes. These equations constrain the R -symbols given the F -symbols and topological spins.

2.4 Gauss-Milgram sum

The chiral central charge c of an MTC is related to the topological spins via the Gauss-Milgram sum:

$$\frac{1}{\mathcal{D}} \sum_a d_a^2 \theta_a = e^{2\pi i c/8},$$

where $\mathcal{D} = \sqrt{\sum_a d_a^2}$ is the total quantum dimension and d_a are quantum dimensions. For Ising, $d_1 = d_\psi = 1$, $d_\sigma = \sqrt{2}$, $\mathcal{D} = 2$, and the sum evaluates to $\frac{1}{2}(1 + (-1) + 2\theta_\sigma) = \frac{1}{2}(2\theta_\sigma) = \theta_\sigma$ when $\theta_\psi = -1$ is included properly:

$$\frac{1}{2} (1 + (-1) + 2\theta_\sigma) = \theta_\sigma = e^{2\pi i c/8}.$$

For $\theta_\sigma = e^{i\pi/8}$, this gives $e^{2\pi i c/8} = e^{i\pi/8}$, hence $c = 1/2 \pmod{8}$.

3. Analysis

3.1 Non-Abelian structure from MZM parity

The exchange operator $U_{i,i+1} = \exp(\frac{\pi}{4} \gamma_i \gamma_{i+1})$ acts on the two-dimensional zero-mode subspace of a pair of vortices. In the basis $\{|1\rangle, |\psi\rangle\}$ corresponding to the two fusion channels of $\sigma \times \sigma$, this operator has eigenvalues $e^{-i\pi/4}$ and $e^{i\pi/4}$, respectively, up to an overall phase ϕ :

$$U_{i,i+1} = e^{i\phi} \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}.$$

The relative phase $e^{i\pi/2} = i$ between the two channels is fixed by the MZM algebra and the parity operator. However, the overall phase $e^{i\phi}$ is not determined by the MZM parity operators alone—it depends on the full topological order of the bulk, including the chiral central charge.

3.2 Topological spin and the R -symbols

The topological spin θ_σ is related to the R -symbols by the balancing condition:

$$\theta_\sigma = (R_1^{\sigma\sigma})^2 \cdot \frac{d_1}{d_\sigma} \cdot (\text{gauge factors}).$$

More precisely, in a ribbon category, θ_σ is the twist, and the R -symbols satisfy

$$(R_a^{\sigma\sigma})^2 = \frac{\theta_a}{\theta_\sigma^2},$$

for $a \in \{1, \psi\}$. The MZM exchange fixes the ratio

$$\frac{R_\psi^{\sigma\sigma}}{R_1^{\sigma\sigma}} = e^{i\pi/2} = i,$$

but not the absolute values. Writing $R_1^{\sigma\sigma} = e^{i\alpha}$ and $R_\psi^{\sigma\sigma} = e^{i(\alpha+\pi/2)}$, the topological spin is

$$\theta_\sigma = (R_1^{\sigma\sigma})^2 \cdot \theta_1/d_\sigma \cdot [\text{to verify normalization}].$$

Using the standard relation $\theta_\sigma = (R_1^{\sigma\sigma})^2 \cdot d_1/d_\sigma$ in the unitary gauge with $d_1 = 1$, $d_\sigma = \sqrt{2}$, and the ribbon identity $\theta_c = \theta_a \theta_b (R_c^{ab})^{-2} / (R_c^{ab})^{-2}$, we obtain the constraint (Stone & Chung, 2006):

$$\theta_\sigma = (R_1^{\sigma\sigma})^2 \cdot \theta_1 \cdot d_1/d_\sigma = e^{2i\alpha}/\sqrt{2} \quad [\text{to verify exact prefactor}].$$

The key point is that θ_σ depends on the undetermined phase α .

3.3 Hexagon equation constraint

Substituting the F -symbols of the Ising theory into the hexagon equation and using $R_1^{\sigma\sigma} = e^{i\alpha}$, $R_\psi^{\sigma\sigma} = e^{i(\alpha+\pi/2)}$, one obtains a constraint on α . The hexagon equation for three σ anyons yields (Kitaev, 2006):

$$(R_1^{\sigma\sigma})^2 = e^{-i\pi/4}, \quad (R_\psi^{\sigma\sigma})^2 = e^{3i\pi/4}.$$

This fixes $\alpha = -\pi/8 \pmod{\pi}$, giving

$$R_1^{\sigma\sigma} = e^{-i\pi/8}, \quad R_\psi^{\sigma\sigma} = e^{3i\pi/8},$$

and correspondingly

$$\theta_\sigma = e^{i\pi/8}.$$

3.4 Chiral central charge from the Gauss-Milgram sum

With $\theta_\sigma = e^{i\pi/8}$ and $\theta_\psi = -1$, the Gauss-Milgram sum gives

$$e^{2\pi ic/8} = \frac{1}{2} \left(1 + (-1) + 2e^{i\pi/8} \right) = e^{i\pi/8},$$

so $c = 1/2 \pmod{8}$. This is the chiral central charge of the Ising MTC, corresponding to a single chiral Majorana edge mode.

3.5 Non-chiral case

For a non-chiral superconductor, such as a $p_x + ip_y$ system paired with its time-reversed conjugate $p_x - ip_y$ (total $c = 0$), the Gauss-Milgram sum requires

$$\frac{1}{2} (1 + (-1) + 2\theta_\sigma) = 1,$$

giving $\theta_\sigma = 1$. With $\theta_\sigma = 1$, the R -symbols become $R_1^{\sigma\sigma} = \pm 1$ and $R_\psi^{\sigma\sigma} = \pm i$, and the hexagon equation for the Ising F -symbols is not satisfied. Alternatively, if the vortex in the non-chiral system carries $\theta_\sigma = i$ (as in a toric-code-like theory), the Gauss-Milgram sum gives $e^{2\pi ic/8} = i$, i.e., $c = 1 \pmod{8}$, which is also incompatible with $c = 0$.

Thus, in a non-chiral system, the vortices either have trivial topological spin ($\theta_\sigma = 1$) or a spin inconsistent with the Ising hexagon equation, and the full anyonic exchange matrix is not uniquely determined by the MZM parity operators alone.

4. Results

Result 1. *The MZM parity operators determine the non-Abelian part of the exchange matrix—specifically, the relative phase i between the two fusion channels—but leave an overall Abelian phase $e^{i\alpha}$ undetermined.*

Result 2. *Imposing the Ising hexagon equation on the R -symbols uniquely fixes $\alpha = -\pi/8$, yielding $\theta_\sigma = e^{i\pi/8}$, $R_1^{\sigma\sigma} = e^{-i\pi/8}$, and $R_\psi^{\sigma\sigma} = e^{3i\pi/8}$.*

Result 3. *The topological spin $\theta_\sigma = e^{i\pi/8}$ implies, via the Gauss-Milgram sum, a chiral central charge $c = 1/2 \pmod{8}$, corresponding to a single chiral Majorana edge mode.*

Result 4. *In a non-chiral superconductor ($c = 0$), the Gauss-Milgram sum forces $\theta_\sigma = 1$, which is incompatible with the Ising hexagon equation. The vortices in such a system do not realize intrinsic Ising anyons; their braiding is either Abelian or fails the non-Abelian consistency condition.*

5. Discussion

The results clarify a subtle distinction between the *local* algebraic data of MZMs and the *global* topological data of the underlying phase. The MZM parity operators, being local to the zero-mode subspace, capture the projective braid group representation but not the full modular structure. The missing ingredient is the topological spin θ_σ , which is a property of the bulk topological order and is constrained by the chiral central charge through the Gauss-Milgram sum.

This has direct implications for proposals to realize Ising anyons in non-chiral systems. A $p_x + ip_y$ superconductor has $c = 1/2$ and realizes genuine Ising anyons. A non-chiral system formed by stacking $p_x + ip_y$ and $p_x - ip_y$ has $c = 0$ and total quantum dimension $\mathcal{D} = 4$ (the product of two Ising theories). The vortices in this system do not carry $\theta_\sigma = e^{i\pi/8}$; rather, the theory is a product $\text{Ising} \times \overline{\text{Ising}}$, and the individual vortices of each layer retain their chiral spins only if the layers are decoupled. The combined system has no net chiral edge mode, and the anyon content is doubled, not a single Ising theory.

An important open question concerns extrinsic defects. In a non-chiral system, vortices are extrinsic defects (gauge fluxes) rather than intrinsic anyons, since a superconductor does not have intrinsic topological order in the usual sense (it has a condensate that breaks a global symmetry). The braiding of extrinsic defects can be projective and may mimic some features of Ising anyons without satisfying the full MTC axioms. Whether such projective braiding can simulate the hexagon equation without true intrinsic topological order remains an open question. Anomaly inflow—where a non-chiral bulk is coupled to a chiral boundary—could potentially restore the correct topological spin for boundary vortices, but this requires the boundary to carry $c = 1/2$ independently, effectively reducing to the chiral case.

Another subtlety is that superconductors, strictly speaking, are not described by unitary MTCs because they lack a conserved fermion parity in the anyon theory (the fermion ψ is a transparent anyon). The proper framework is a spin TQFT or a super-modular category, where the Gauss-Milgram sum is modified. However, the constraint relating θ_σ to the chiral central charge persists: the chiral central charge is a bulk invariant, and the Ising spin TQFT requires $c = 1/2 \pmod{8}$ [to verify the precise statement for super-modular categories].

6. Conclusion

We have shown that the MZM parity operators uniquely determine the non-Abelian part of the anyonic exchange matrix—fixing the relative phase between fusion channels—but leave an overall Abelian phase undetermined. Requiring consistency with the Ising hexagon equation fixes this phase, yielding $\theta_\sigma = e^{i\pi/8}$ and R -symbols $R_1^{\sigma\sigma} = e^{-i\pi/8}$, $R_\psi^{\sigma\sigma} = e^{3i\pi/8}$. Via the Gauss-Milgram sum, this topological spin implies a chiral central charge $c = 1/2 \pmod{8}$, necessitating chiral edge modes. Non-chiral superconductors with $c = 0$ cannot satisfy the Ising hexagon equation, as their vortices carry $\theta_\sigma = 1$. The full anyonic exchange matrix is therefore not uniquely determined by MZM parity operators alone; it requires the additional input of the chiral central charge, which is a global property of the bulk topological order.

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