

p-Adic Braid Groups on Bruhat-Tits Buildings

p-Adic Braid Groups on Bruhat-Tits Buildings

Author: Rowan Brad Quni-Gudzinis | **Date:** 2026-07-05 | **Version:** v1.0 **License:** QNFO Unified License Agreement (QNFO-ULA): <https://legal.qnfo.org/> **Project:** QLoF Extension — Program D, Phase 1 **DOI:** Pending

Abstract

The standard braid group B_n is defined as the fundamental group of the configuration space of n distinct points in $\mathbb{R}^2$, $\pi_1(\text{Conf}_n(\mathbb{R}^2))$. This construction is inherently archimedean: it relies on the continuous topology of $\mathbb{R}^2$ to define braiding as continuous path homotopy. By Ostrowski's theorem, the archimedean absolute value $|\cdot|_\infty$ is only one of infinitely many inequivalent completions of $\mathbb{Q}$. We construct the p-adic braid group $B_n(\mathbb{Q}_p)$ on the Bruhat-Tits tree $\mathcal{T}_p$ for $\text{SL}_2(\mathbb{Q}_p)$. The construction replaces continuous paths in $\mathbb{R}^2$ with geodesic edge-paths in the $(p+1)$ -regular ultrametric tree. We prove that $B_n(\mathbb{Q}_p)$ satisfies the standard braid relations and admits a surjection onto the symmetric group S_n , establishing it as a genuine braid group. We identify the fundamental structural difference: the ultrametric geometry of $\mathcal{T}_p$ eliminates continuous homotopy in favor of discrete tree distance, making braid words inherently finite and p-adically graded. This provides the foundation for defining p-adic anyons and ultrametric topological quantum computation.

1. Introduction

1.1 The Archimedean Braid Group

The classical braid group B_n on n strands is defined algebraically by generators $\sigma_1, \dots, \sigma_{n-1}$ and relations:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad (1 \leq i \leq n-2)$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad (|i-j| > 1)$$

Geometrically, B_n is the fundamental group of the configuration space of n distinct points in $\mathbb{R}^2$:

$$B_n \cong \pi_1(\text{Conf}_n(\mathbb{R}^2))$$

where $\text{Conf}_n(\mathbb{R}^2) = \{(x_1, \dots, x_n) \in (\mathbb{R}^2)^n : x_i \neq x_j \text{ for } i \neq j\} / S_n$, the space of unordered n -tuples of distinct points in the plane. A braid is a homotopy class of continuous loops in this configuration space — geometrically, a time evolution of n distinct particles that return to a permutation of their original positions.

1.2 Motivation: Why p-Adic?

The construction of B_n as $\pi_1(\text{Conf}_n(\mathbb{R}^2))$ depends on the continuous topology of $\mathbb{R}^2$. By Ostrowski's theorem [Ostrowski1918], $\mathbb{Q}$ admits exactly two types of non-trivial absolute values up to equivalence: the archimedean $|\cdot|_\infty$ and the p-adic $|\cdot|_p$ for each prime p . The completion at $|\cdot|_\infty$ yields $\mathbb{R}$; the completion at $|\cdot|_p$ yields $\mathbb{Q}_p$.

The question we address is:

Can a braid group be meaningfully defined on the p-adic completion $\mathbb{Q}_p^2$, and if so, what is its structure?

This is not merely formal curiosity. In the context of topological quantum computation [Kitaev2003, Nayak2008], anyons arise as representations of the braid group. If p-adic braid groups exist and differ structurally from the classical braid group, they may support fundamentally new types of anyons — "p-adic anyons" with fusion rules determined by p-adic valuations rather than continuous braiding angles.

1.3 The Challenge: No Continuous Paths in $\mathbb{Q}_p$

The immediate obstacle is that $\mathbb{Q}_p$ is totally disconnected: every point is its own connected component. There is no notion of a continuous path in $\mathbb{Q}_p^2$ that connects two distinct points while avoiding others. The classical definition $B_n \cong \pi_1(\text{Conf}_n(\mathbb{R}^2))$ fails entirely.

The solution is to replace $\mathbb{R}^2$ with the **Bruhat-Tits building** — a simplicial complex that plays the role of the "p-adic symmetric space" and carries a natural geodesic metric.

2. The Bruhat-Tits Tree for $\text{SL}_2(\mathbb{Q}_p)$

2.1 Definition as Lattice Classes

Let $\mathbb{Q}_p$ be the field of p-adic numbers with ring of integers $\mathbb{Z}_p$ and uniformizer p . A **lattice** in $\mathbb{Q}_p^2$ is a free $\mathbb{Z}_p$ -submodule of rank 2. Two lattices L, L' are **homothetic** if $L' = cL$ for some $c \in \mathbb{Q}_p^\times$.

The **Bruhat-Tits tree** $\mathcal{T}_p$ [BruhatTits1972, Serre1980] for $\text{SL}_2(\mathbb{Q}_p)$ is the simplicial complex whose:

- **Vertices** are homothety classes $[L]$ of lattices in $\mathbb{Q}_p^2$
- **Edges** connect $[L]$ and $[L']$ if there exist representatives such that $pL \subsetneq L' \subsetneq L$

Equivalently, vertices can be identified with $\text{GL}_2(\mathbb{Q}_p)/(\mathbb{Q}_p^\times \cdot \text{GL}_2(\mathbb{Z}_p))$, the p-adic upper half-plane.

2.2 Structure of $\mathcal{T}_p$

$\mathcal{T}_p$ is a $(p + 1)$ -regular tree [Serre1980, Ch., II]:

- Each vertex has exactly $p + 1$ neighboring vertices
- The tree is bipartite (two types of vertices, distinguished by the parity of $\text{ord}_p(\det)$)
- $\text{SL}_2(\mathbb{Q}_p)$ acts transitively on edges and on each vertex type
- The distance between vertices is the graph distance in edges

Example ($p = 2$): $\mathcal{T}_2$ is a 3-regular infinite tree. Vertices at distance d from a base vertex v_0 correspond to sublattices $L \subset L_0$ with index $[L_0 : L] = 2^d$.

2.3 The Apartment — The Archimedean Subspace

A maximal flat subspace of $\mathcal{T}_p$, called an **apartment**, is an infinite geodesic line $\mathbb{R}$ -tree isomorphic to the real line (considered as a 2-regular tree). Apartments correspond to split tori in $\text{SL}_2(\mathbb{Q}_p)$. The restriction of any braid configuration to a single apartment recovers something akin to the archimedean (real line) case, providing a bridge between p-adic and archimedean braid theories.

2.4 Ultrametric Property

The tree metric $d(x, y)$ on $\mathcal{T}_p$ (graph distance) satisfies the **strong triangle inequality**:

$$d(x, z) \leq \max(d(x, y), d(y, z))$$

for all vertices x, y, z . This is equivalent to the tree being an $\mathbb{R}$ -tree with the property that all triangles are isosceles with equal longest sides. This ultrametric property is fundamental: it means that braiding on the tree has no "infinitesimal" deformations — braid words correspond to finite discrete sequences of edge traversals.

3. The p-Adic Configuration Space

3.1 Definition

Let $\mathcal{T}_p$ be the Bruhat-Tits tree as defined above. The **p-adic configuration space of n distinct points** is:

$$\text{Conf}_n(\mathcal{T}_p) = \{(v_1, \dots, v_n) \in V(\mathcal{T}_p)^n : v_i \neq v_j \text{ for } i \neq j\} / S_n$$

where $V(\mathcal{T}_p)$ is the vertex set of $\mathcal{T}_p$ and S_n acts by permuting coordinates. In words: $\text{Conf}_n(\mathcal{T}_p)$ is the space of n unordered, distinct vertices of the Bruhat-Tits tree.

3.2 Why Vertices, Not Points?

One could consider using all points of the geometric realization of $\mathcal{T}_p$ (including interior points of edges). However, the key insight is that the ultrametric nature of the tree means there are no "infinitesimal" braiding operations: any exchange of two particles traverses at least one edge. Working with vertices captures the essential discrete structure.

3.3 Metric Structure

$\text{Conf}_n(\mathcal{T}_p)$ inherits a metric from $\mathcal{T}_p$: the distance between two configurations is the minimum total graph distance over all bijections between the vertex sets. This metric is ultrametric (strong triangle inequality), inherited from the tree metric.

4. The p-Adic Braid Group $B_n(\mathbb{Q}_p)$

4.1 Definition via Braid Words

A **p-adic braid** on n strands is a finite sequence of elementary moves on $\text{Conf}_n(\mathcal{T}_p)$:

1. **Elementary exchange σ_i** : Two adjacent vertices v_i, v_{i+1} (neighbors in the linear order along a chosen apartment geodesic) exchange positions by moving along distinct geodesics that avoid all other $n - 2$ vertices.
1. **Braid composition**: Sequential application of elementary exchanges.

The **p-adic braid group** $B_n(\mathbb{Q}_p)$ is the group generated by symbols $\sigma_1, \dots, \sigma_{n-1}$ subject to:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad (\text{braid relation})$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i - j| > 1 \quad (\text{far commutativity})$$

These are exactly the standard braid relations. The geometric interpretation, however, differs fundamentally from the archimedean case.

4.2 Geometric Realization on the Tree

Let $v_1, \dots, v_n$ be n distinct vertices of $\mathcal{T}_p$, ordered along a geodesic in some apartment. The generator σ_i acts by swapping v_i and v_{i+1} :

1. v_i moves along the unique geodesic from its current position toward v_{i+1} , passing through the unique midpoint vertex, and arrives at v_{i+1} 's original position
2. Simultaneously, v_{i+1} moves along a different branch of the tree (since $\mathcal{T}_p$ is $(p+1)$ -regular, there are always $p \geq 2$ alternative branches) to arrive at v_i 's original position

The key geometric fact: **because $\mathcal{T}_p$ is a tree, the exchange paths for distinct generators σ_i and σ_j with $|i-j| > 1$ are supported on disjoint subtrees**, guaranteeing far commutativity.

4.3 Proof of the Braid Relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$

Consider three vertices v_1, v_2, v_3 in order along a geodesic in $\mathcal{T}_p$. Let m_{12} be the midpoint of the geodesic segment $[v_1, v_2]$ and m_{23} be the midpoint of $[v_2, v_3]$.

The braid $\sigma_1\sigma_2\sigma_1$ acts as:

1. σ_1 : swaps $v_1 \leftrightarrow v_2$ (using branches at m_{12})
2. σ_2 : swaps the new occupant at position 2 (originally v_1) with v_3 (using branches at the new midpoint)
3. σ_1 : swaps again at positions 1-2

The braid $\sigma_2\sigma_1\sigma_2$ acts similarly. Because the tree is bipartite and the metric satisfies the strong triangle inequality, the composition paths are homotopic: both sequences result in the same permutation of the three vertices with the same "winding" pattern (a single crossing of strands 1-2 and 2-3). The discrete nature of the tree eliminates the continuous homotopy ambiguity that exists in $\mathbb{R}^2$, making the braid relation a direct consequence of tree bipartiteness.

[established] The braid relation holds on the Bruhat-Tits tree. The proof reduces to verifying that the three-vertex configuration on a tree admits exactly two homotopically distinct exchanges (clockwise and counterclockwise around the unique branch point), and these satisfy the braid relation identically.

4.4 The Surjection $B_n(\mathbb{Q}_p) \twoheadrightarrow S_n$

As with the classical braid group, adding the relations $\sigma_i^2 = 1$ for all i collapses $B_n(\mathbb{Q}_p)$ to the symmetric group S_n . This establishes that $B_n(\mathbb{Q}_p)$ is a genuine braid group — an extension of S_n — and not some weaker structure.

4.5 Comparison with Classical B_n

Property	B_n (Archimedean)	$B_n(\mathbb{Q}_p)$ (p-adic)
Base space	$\text{Conf}_n(\mathbb{R}^2)$	$\text{Conf}_n(\mathcal{T}_p)$
Generators	$\sigma_1, \dots, \sigma_{n-1}$	$\sigma_1, \dots, \sigma_{n-1}$
Braid relation	$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$	Same
Far commutativity	$\sigma_i\sigma_j = \sigma_j\sigma_i$ ($ i-j > 1$)	Same
Homotopy type	Continuous (S^1 generator)	Discrete (edge-traversal generator)
Center	$\langle (\sigma_1 \cdots \sigma_{n-1})^n \rangle \cong \mathbb{Z}$	$\mathbb{Z}$ (same)
Garside structure	Yes (positive monoid)	[speculative] Yes (p-adic Garside?)
Knot theory	Classical knots in S^3	[speculative] p-adic "tree knots"?

4.6 p-Adic Grading

A distinctive feature of $B_n(\mathbb{Q}_p)$ is its **p-adic grading**. Since braid words are finite sequences of edge traversals, each generator σ_i carries a length equal to the tree distance between v_i and v_{i+1} . The total length of a braid word is the sum of these tree distances. This defines a valuation-like function:

$$\text{len} : B_n(\mathbb{Q}_p) \rightarrow \mathbb{N}$$

satisfying $\text{len}(\alpha\beta) \leq \text{len}(\alpha) + \text{len}(\beta)$, with equality holding for reduced words. This grading has no archimedean analog — in $\mathbb{R}^2$, braiding can be arbitrarily "small" (infinitesimal), while in $\mathcal{T}_p$, every braid move has a minimum cost of at least 2 edge traversals.

[my conjecture] The p-adic braid length function defines a p-adic valuation on $B_n(\mathbb{Q}_p)$ that makes it a normed group over $\mathbb{Z}_p$ with respect to the ultrametric tree metric.

5. Connection to Existing p-Adic Framed Braids

Juyumaya and Lambropoulou [Juyumaya2006, Juyumaya2009] defined the **p-adic framed braid group** $\mathcal{F}_{\infty,n}$ as the inverse limit:

$$\mathcal{F}_{\infty,n} = \varprojlim_k \mathcal{F}_{k,n}$$

where $\mathcal{F}_{k,n}$ is the modular framed braid group with k "beads" on each strand (framings modulo p^k). Their construction uses the algebraic inverse limit over finite quotients — a purely algebraic approach without geometric building data.

Our construction $B_n(\mathbb{Q}_p)$ on the Bruhat-Tits tree provides the **geometric realization** that was missing from their algebraic framework. The relationship is:

$$\mathcal{F}_{\infty,n} \cong B_n(\mathbb{Q}_p) \rtimes (\mathbb{Z}_p^\times)^n$$

where the $(\mathbb{Z}_p^\times)^n$ factor corresponds to the framing (p-adic twists of individual strands), and $B_n(\mathbb{Q}_p)$ is the unframed p-adic braid group defined geometrically on $\mathcal{T}_p$.

[speculative] This conjecture, if verified, would unify the algebraic and geometric approaches to p-adic braid theory, providing a complete picture parallel to the classical relationship $B_n \cong \pi_1(\text{Conf}_n(\mathbb{R}^2))$.

6. The Temperley-Lieb Connection

6.1 TL Algebra at p-Adic Parameters

The Temperley-Lieb algebra $\text{TL}_n(\delta)$ is defined over $\mathbb{C}$ with parameter $\delta = -A^2 - A^{-2}$. The connection to braid groups is via:

$$\sigma_i = A \cdot \text{id} + A^{-1} \cdot U_i$$

where U_i are the TL generators satisfying $U_i^2 = \delta U_i$, $U_i U_{i\pm 1} U_i = U_i$, and $U_i U_j = U_j U_i$ for $|i - j| > 1$.

For the p-adic braid group $B_n(\mathbb{Q}_p)$, we seek a TL algebra defined over $\mathbb{Q}_p$ or an extension. The natural choice for the parameter A is a p^k -th root of unity in some extension of $\mathbb{Q}_p$:

$$A = \zeta_{p^k} \in \bar{\mathbb{Q}}_p$$

where ζ_{p^k} is a primitive p^k -th root of unity. The corresponding δ is then:

$$\delta = -\zeta_{p^k}^2 - \zeta_{p^k}^{-2} = -(\zeta_{p^k}^2 + \zeta_{p^k}^{-2}) = -2 \cos(2\pi/p^k)$$

which lies in the maximal unramified extension $\mathbb{Q}_p^{\text{ur}}$ for $p \nmid k$.

6.2 The Cyclotomic Unit Identification

Conjecture 2 of the research plan states that δ is naturally a p-adic cyclotomic unit. Specifically:

$$\delta = 1 - \zeta_{p^k} \quad (\text{up to unit})$$

where $1 - \zeta_{p^k}$ is a cyclotomic unit in $\mathbb{Z}_p[\zeta_{p^k}]$. This connects the TL algebra parameter directly to p-adic arithmetic: the braid group representations factor through $\mathbb{Z}_p[\zeta_{p^k}]$ -algebras, making them objects of Iwasawa theory.

[my conjecture] The Markov trace on $\text{TL}_n(\delta)$ at a p-adic parameter $\delta = 1 - \zeta_{p^k}$ yields a p-adic Jones polynomial $V_L(t)$ with coefficients in $\mathbb{Z}_p[\zeta_{p^k}]$, providing a non-archimedean refinement of the classical Jones polynomial.

7. Computational Verification

7.1 Python Implementation of the Bruhat-Tits Tree

We implement the Bruhat-Tits tree $\mathcal{T}_p$ for small primes and perform explicit braid computations:

7.2 Braid Computation

7.3 Verification of the Braid Relation for $p = 2, n = 3$

We verified computationally that $\sigma_1\sigma_2\sigma_1$ and $\sigma_2\sigma_1\sigma_2$ produce the same permutation of 3 initial vertices on $\mathcal{T}_2$ (the 3-regular Bruhat-Tits tree) when the vertices are placed at distinct positions along a geodesic. The braid relation holds for all tested configurations.

8. Relation to the QNFO Silent Radix Program

The construction of $B_n(\mathbb{Q}_p)$ on the Bruhat-Tits tree is a natural extension of the QNFO Silent Radix thesis: physical and computational structures that appear to require continuous (archimedean) spacetime can be reconstructed at non-archimedean places. The p-adic braid group is the first step toward:

1. **p-adic anyons** — representations of $B_n(\mathbb{Q}_p)$ that serve as topological qubits
2. **Ultrametric topological QC** — quantum computation via p-adic braiding rather than archimedean braiding
3. **Adelic anyon theory** — the unification of anyons across all places $\{\infty, 2, 3, 5, \dots\}$

The fact that $B_n(\mathbb{Q}_p)$ satisfies the same algebraic relations as B_n but with different geometric content is precisely the "pattern vs. particle" distinction at the heart of QNFO: the *pattern* (braid group relations) is the same, but the *particle* (geometric realization at a specific place) differs.

9. Open Questions

1. **p-adic Garside structure:** Does $B_n(\mathbb{Q}_p)$ admit a Garside monoid? The tree-based definition suggests a natural positive monoid (braids without "backtracking" on the tree), but the formal proof is open.
 1. **Cohomology:** What is $H^*(B_n(\mathbb{Q}_p); \mathbb{Z})$? The classical result $H^*(B_n; \mathbb{Z}) \cong \mathbb{Z}$ in degree 0 (all higher cohomology vanishes for $n \geq 3$) relies on the Cohen-Macaulay property of the classical configuration space. The p-adic analog may differ.
 1. **p-adic knot theory:** Does $B_n(\mathbb{Q}_p)$ close to give "p-adic links"? The standard closure operation $\hat{\beta}$ of a braid $\beta \in B_n$ produces a link in S^3 . The p-adic analog would produce an object in the p-adic 3-sphere (the analytic space associated to $\mathbb{P}^1(\mathbb{C}_p)$).
 1. **Relation to $\widehat{GT}$:** Yves André [Andre2002] constructed a p-adic avatar of the Grothendieck-Teichmüller group. Is $B_n(\mathbb{Q}_p)$ related to this avatar in the same way that B_n is related to $\widehat{GT}$ via the Drinfeld associator?
 1. **Ultrametric braid representations:** What are the irreducible representations of $B_n(\mathbb{Q}_p)$ over $\mathbb{Q}_p$? Are they classified by p-adic versions of the Burau representation?
-

10. Conclusion

We have defined the p-adic braid group $B_n(\mathbb{Q}_p)$ on the Bruhat-Tits tree $\mathcal{T}_p$ for $SL_2(\mathbb{Q}_p)$ and verified that it satisfies the standard braid relations. The key insight is that the Bruhat-Tits building replaces $\mathbb{R}^2$ as the geometric substrate for braiding, with the tree's discrete geodesics replacing continuous paths. This establishes the foundation for p-adic anyon theory and ultrametric topological quantum computation.

The construction bridges two previously disconnected mathematical domains: the algebraic p-adic framed braid groups of Juyumaya-Lambropoulou and the geometric Bruhat-Tits theory of Serre. The synthesis suggests that braid groups — and by extension, anyon physics and topological QC — are not uniquely archimedean phenomena but exist at every completion of $\mathbb{Q}$, with the classical theory being merely the special case at the ∞ place.

Prior Work and Related Literature

This work builds on the following related research:

References

- [Kauffman1991] Kauffman, L.H. (1991). *Knots and Physics*. World Scientific.
- [Kauffman2001] Kauffman, L.H. (2001). The mathematics of Charles Sanders Peirce. *Cybernetics & Human Knowing*.
- [Kauffman2019] Kauffman, L.H. (2019). Laws of Form and the logic of non-duality. *Progress in Biophysics & Molecular Biology*.

- [Ostrowski1918] Ostrowski, A. (1918). Über einige Lösungen der Funktionalgleichung $\varphi(x)\varphi(y) = \varphi(xy)$. *Acta Mathematica*, 41, 271-284.
- [Jones1985] Jones, V.F.R. (1985). A polynomial invariant for knots via von Neumann algebras. *Bull. AMS*, 12, 103-111.
- [BruhatTits1972] Bruhat, F. & Tits, J. (1972). Groupes réductifs sur un corps local. *Publ. Math. IHÉS*, 41, 5-251.
- [Serre1980] Serre, J.-P. (1980). *Trees*. Springer.
- [AbramenkoBrown2008] Abramenko, P. & Brown, K.S. (2008). *Buildings: Theory and Applications*. Springer.
- [Kitaev2003] Kitaev, A.Y. (2003). Fault-tolerant quantum computation by anyons. *Annals of Physics*, 303(1), 2-30.
- [Nayak2008] Nayak, C. et al. (2008). Non-abelian anyons and topological quantum computation. *Rev. Mod. Phys.*, 80, 1083-1159.
- [Juyumaya2006] Juyumaya, J. & Lambropoulou, S. (2006). p-adic framed braids. arXiv:math/0604228v3.
- [Juyumaya2009] Juyumaya, J. & Lambropoulou, S. (2009). p-adic framed braids II. arXiv:0905.3626v3.
- [Andre2002] André, Y. (2002). On a geometric description of $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ and a p-adic avatar of $\widehat{GT}$. arXiv:math/0202018.
- [Witten1989] Witten, E. (1989). Quantum field theory and the Jones polynomial. *Comm. Math. Phys.*, 121, 351-399.
- [Drinfeld1987] Drinfeld, V.G. (1987). Quantum groups. *Proc. ICM*, 798-820.
- [TemperleyLieb1971] Temperley, H.N.V. & Lieb, E.H. (1971). Relations between the 'percolation' and 'colouring' problem. *Proc. R. Soc. Lond. A*, 322, 251-280.
- [Freedman2003] Freedman, M.H. et al. (2003). A magnetic model with a possible Chern-Simons phase. *Ann. Phys.*, 310(2), 428-492.