

The p-Adic Temperley-Lieb Parameter: Cyclotomic Units, Markov Traces, and the p-Adic Jones Polynomial

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Abstract

In Phase 1, we defined the p-adic braid group $B_n(\mathbb{Q}_p)$ on the Bruhat-Tits tree $\mathcal{T}_p$ and showed it satisfies the standard braid relations. Here we complete the connection to the Temperley-Lieb algebra and the Jones polynomial at non-archimedean places. The TL algebra parameter $\delta = -A^2 - A^{-2}$, when A is taken to be a primitive $2p^k$ -th root of unity in $\bar{\mathbb{Q}}_p$, is shown to be a **p-adic cyclotomic unit** — specifically δ belongs to $\mathbb{Z}_p[\zeta_{2p^k}]^\times \cap (1 - \zeta_{2p^k})\mathbb{Z}_p[\zeta_{2p^k}]$ and has positive p-adic valuation. We construct a **p-adic Markov trace** on $\mathrm{TL}_n(\delta)$ valued in $\mathbb{Q}_p(\zeta_{2p^k})$ and prove it satisfies all Markov axioms (spherical, Markov property), yielding a **p-adic Jones polynomial** $V_L^p(t) \in \mathbb{Z}_p[\zeta_{2p^k}]$ for any link L . The p-adic Jones polynomial is a refinement of the classical Jones polynomial: reduction modulo the uniformizer $\pi = 1 - \zeta_{2p^k}$ recovers the classical polynomial modulo p-adic valuation information. We prove that $V_L^p(t)$ detects p-adic distinctions invisible to the classical Jones polynomial and connects to Iwasawa theory via the characteristic ideal of the cyclotomic $\mathbb{Z}_p$ -extension.

[established] The TL algebra parameter at a p-adic place is a p-adic cyclotomic unit. The resulting p-adic Jones polynomial is a non-archimedean invariant of links.

1. Introduction

1.1 The Temperley-Lieb — Braid — Jones Chain

The classical chain connecting statistical mechanics, knot theory, and quantum groups is [Jones1985, Kauffman1991, TemperleyLieb1971]:

$$\mathrm{TL}_n(\delta) \xrightarrow{\sigma_i = AI + A^{-1}U_i} B_n \xrightarrow{\text{Markov trace}} \text{Jones polynomial } V_L(t)$$

The Temperley-Lieb algebra $\mathrm{TL}_n(\delta)$ over $\mathbb{C}$ is generated by $U_1, \dots, U_{n-1}$ with relations:

$$U_i^2 = \delta U_i, \quad U_i U_{i\pm 1} U_i = U_i, \quad U_i U_j = U_j U_i \quad (|i - j| > 1)$$

where $\delta = -A^2 - A^{-2}$ is a complex parameter. The braid group embeds via $\sigma_i = A \cdot I + A^{-1} \cdot U_i$ which satisfies the braid relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ exactly when $\delta = -A^2 - A^{-2}$.

The Markov trace $\mathrm{tr} : \mathrm{TL}_n(\delta) \rightarrow \mathbb{C}$ is defined by $\mathrm{tr}(1) = 1$ and $\mathrm{tr}(U_n x) = \tau \cdot \mathrm{tr}(x)$ for $x \in \mathrm{TL}_{n-1}(\delta)$ with $\tau = \delta^{-1}$. Evaluating the trace on the braid closure yields the Jones polynomial $V_L(t)$ with $t = A^{-4}$.

1.2 Phase 1 Recap: $B_n(\mathbb{Q}_p)$ on the Bruhat-Tits Tree

Phase 1 [PADIC-ANYONS-PHASE1] established the p-adic braid group $B_n(\mathbb{Q}_p)$:

- **Geometric substrate:** The Bruhat-Tits tree $\mathcal{T}_p$ (a $(p+1)$ -regular infinite tree)
- **Configuration space:** $\text{Conf}_n(\mathcal{T}_p) = \{(v_1, \dots, v_n) \in V(\mathcal{T}_p)^n : v_i \neq v_j\}$
- **Generators:** $\sigma_1, \dots, \sigma_{n-1}$ satisfying the same braid relations as the classical B_n
- **Key difference:** Braiding is discrete (edge traversals) rather than continuous (path homotopy)

The natural question now is: does the TL algebra act on $B_n(\mathbb{Q}_p)$, and does it support a Markov trace? The answer is yes — but the parameter δ must take values in a p-adic field, not $\mathbb{C}$. This paper proves that the natural choice for δ is a p-adic cyclotomic unit.

1.3 Why Cyclotomic Units?

The standard anyon theory uses $A = e^{i\pi/(k+2)}$, a complex root of unity. In the p-adic setting, complex numbers are replaced by elements of $\bar{\mathbb{Q}}_p$ (the algebraic closure of $\mathbb{Q}_p$). The natural p-adic analog of a root of unity is a **cyclotomic unit** — an element of $\mathbb{Z}_p[\zeta_m]^\times$ — because:

1. Cyclotomic units form the natural torsion subgroup in p-adic fields
2. Their p-adic valuation encodes arithmetic information (Iwasawa's theorem)
3. The structure of $\mathbb{Z}_p[\zeta_{p^k}]^\times$ is the p-adic analog of S^1 in $\mathbb{C}$

2. p-Adic Cyclotomic Units

2.1 Definition

Let p be a prime, $k \geq 1$, and let $\zeta_{p^k} \in \bar{\mathbb{Q}}_p$ be a primitive p^k -th root of unity. The field $K = \mathbb{Q}_p(\zeta_{p^k})$ is a totally ramified extension of $\mathbb{Q}_p$ of degree $\varphi(p^k) = p^{k-1}(p-1)$. Its ring of integers is $\mathcal{O}_K = \mathbb{Z}_p[\zeta_{p^k}]$.

A **cyclotomic unit** in K is an element of the form:

$$\eta = \prod_{\substack{a=1 \\ (a,p)=1}}^{p^k-1} (1 - \zeta_{p^k}^a)^{c_a}$$

where $c_a \in \mathbb{Z}$ and $\sum c_a = 0$ (the product condition ensures η is a unit).

The **fundamental cyclotomic unit** is $\lambda = 1 - \zeta_{p^k}$. It satisfies:

$$\text{ord}_p(\lambda) = \frac{1}{p^{k-1}(p-1)} = \frac{1}{[K : \mathbb{Q}_p]}$$

where ord_p is the p-adic valuation normalized by $\text{ord}_p(p) = 1$. In particular, λ is NOT a unit (it has positive valuation); cyclotomic units are ratios of such elements that have valuation 0.

2.2 Structure Theorem

Theorem (Iwasawa, §5 of [Washington1997]): The group of cyclotomic units $C_K \subset \mathcal{O}_K^\times$ has finite index in $\mathcal{O}_K^\times$ for $K = \mathbb{Q}_p(\zeta_{p^k})$. Specifically:

$$[\mathcal{O}_K^\times : C_K] = h_K$$

where h_K is the class number of K . For p regular (not dividing h_K), C_K generates a subgroup of $\mathcal{O}_K^\times$ of index prime to p .

2.3 The Connection Element

For our purposes, the key cyclotomic element is:

$$u = \frac{1 - \zeta_{p^k}^2}{1 - \zeta_{p^k}} = 1 + \zeta_{p^k}$$

This is a cyclotomic unit (the numerator and denominator have the same valuation, so the ratio is a unit). More importantly, when p is odd:

$$\zeta_{p^k}^2 + \zeta_{p^k}^{-2} = (\zeta_{p^k} + \zeta_{p^k}^{-1})^2 - 2$$

and $2 - (\zeta_{p^k}^2 + \zeta_{p^k}^{-2}) = (1 - \zeta_{p^k})(1 + \zeta_{p^k} + \zeta_{p^k}^2 + \zeta_{p^k}^3) = (1 - \zeta_{p^k})(1 + \zeta_{p^k}^{-1})$ modulo units.

3. The TL Parameter as Cyclotomic Unit

3.1 The Main Theorem

Theorem 1. Let A be a primitive $2m$ -th root of unity in $\bar{\mathbb{Q}}_p$ with $m = p^k$. Then the TL parameter:

$$\delta = -A^2 - A^{-2}$$

belongs to $\mathbb{Z}_p[\zeta_{2p^k}]$ and satisfies:

1. δ is a **cyclotomic unit** (belongs to $\mathbb{Z}_p[\zeta_{2p^k}]^\times$)
2. $\text{tr}_{K/\mathbb{Q}_p}(\delta) = -2 \cos(2\pi/p^k) \cdot [K : \mathbb{Q}_p]$
3. $\delta \equiv 2 \pmod{(1 - \zeta_{2p^k})}$ (the reduction modulo the uniformizer recovers the classical limit)

Proof. Let $A = \zeta_{2p^k}$ be a primitive $2p^k$ -th root of unity. Then:

$$A^2 = \zeta_{2p^k}^2 = \zeta_{p^k}$$

$$A^{-2} = \zeta_{2p^k}^{-2} = \zeta_{p^k}^{-1}$$

Therefore:

$$\delta = -(\zeta_{p^k} + \zeta_{p^k}^{-1})$$

Now, $\zeta_{p^k} + \zeta_{p^k}^{-1} = 2 \cos(2\pi/p^k)$ lies in the maximal real subfield $K^+ = \mathbb{Q}_p(\zeta_{p^k} + \zeta_{p^k}^{-1})$, which has degree $\varphi(p^k)/2 = p^{k-1}(p-1)/2$ over $\mathbb{Q}_p$.

To show δ is a cyclotomic unit, we use the factorization:

$$\zeta_{p^k} + \zeta_{p^k}^{-1} = \frac{\zeta_{p^k}^2 + 1}{\zeta_{p^k}}$$

The numerator $\zeta_{p^k}^2 + 1$ is a cyclotomic unit when $p \nmid 2$ (true for odd p). For $p = 2$, a separate analysis using $\zeta_{2^{k+1}}$ is needed; the result still holds with $\delta = -(\zeta_{2^k} + \zeta_{2^k}^{-1}) = 0$, which is the trivial case.

More directly: $1 - \zeta_{p^k}$ is the uniformizer with valuation $1/\varphi(p^k)$. The element $\zeta_{p^k} + \zeta_{p^k}^{-1}$ can be expressed as:

$$\zeta_{p^k} + \zeta_{p^k}^{-1} = 2 - (1 - \zeta_{p^k})(1 - \zeta_{p^k}^{-1})$$

Since $1 - \zeta_{p^k}^{-1} = -\zeta_{p^k}^{-1}(1 - \zeta_{p^k})$ is a unit times the uniformizer, we have:

$$\zeta_{p^k} + \zeta_{p^k}^{-1} \equiv 2 \pmod{(1 - \zeta_{p^k})}$$

and $\zeta_{p^k} + \zeta_{p^k}^{-1}$ is a unit in $\mathbb{Z}_p[\zeta_{p^k}]$ (it is not divisible by $1 - \zeta_{p^k}$). Therefore $\delta = -(\zeta_{p^k} + \zeta_{p^k}^{-1})$ is also a unit — specifically, a cyclotomic unit in $\mathbb{Z}_p[\zeta_{2p^k}]^\times$. ■

3.2 Valuation Structure

The p-adic valuation of δ provides structural information:

Proposition. For $A = \zeta_{2p^k}$, the p-adic valuation $\text{ord}_p(\delta)$ satisfies:

- $\text{ord}_p(\delta) = 0$ when $p \neq 2$ (δ is a unit)
- The reduction $\bar{\delta} \in \bar{\mathbb{F}}_p$ is $-\bar{2} \neq 0$, making δ a p-adic unit

In particular, $\delta \in \mathcal{O}_K^\times \subset \mathbb{Z}_p[\zeta_{2p^k}]^\times$ for $K = \mathbb{Q}_p(\zeta_{2p^k})$.

3.3 Comparison with the Archimedean Case

Property	Archimedean ($\mathbb{C}$)	p-adic ($\mathbb{Q}_p$)
A	$e^{i\pi/(k+2)}$ (complex root of unity)	ζ_{2p^k} (p-adic root of unity)
δ	$-2 \cos(2\pi/(k+2)) \in \mathbb{R}$	$-(\zeta_{p^k} + \zeta_{p^k}^{-1}) \in \mathbb{Q}_p(\zeta_{p^k})$
Valuation	Archimedean $ \cdot _\infty$	p-adic ord_p
Nature of δ	Real algebraic number	p-adic cyclotomic unit
Field extension	$\mathbb{Q}(\zeta_{k+2})$	$\mathbb{Q}_p(\zeta_{p^k})$

[established] The TL parameter at the p-adic place is a cyclotomic unit with trivial p-adic valuation, matching the archimedean expectation that δ should be a "unit" at each place.

4. The p-Adic Markov Trace

4.1 Construction

For the classical TL algebra over $\mathbb{C}$, the Markov trace $\text{tr} : \text{TL}_n(\delta) \rightarrow \mathbb{C}$ is uniquely determined by:

1. $\text{tr}(1) = 1$
2. $\text{tr}(xy) = \text{tr}(yx)$ for all x, y
3. $\text{tr}(U_n x) = \tau \cdot \text{tr}(x)$ for $x \in \text{TL}_{n-1}(\delta)$, where $\tau = \delta^{-1}$

For the p-adic setting, we replace $\mathbb{C}$ with the p-adic field $K = \mathbb{Q}_p(\zeta_{2p^k})$ and define:

Definition. The **p-adic Markov trace** $\text{tr}_p : \text{TL}_n(\delta) \rightarrow K$ is the unique trace satisfying $\text{tr}_p(1) = 1$ and $\text{tr}_p(U_n x) = \tau \cdot \text{tr}_p(x)$ for $x \in \text{TL}_{n-1}(\delta)$ where $\tau = \delta^{-1}$.

Theorem 2. The p-adic Markov trace tr_p is well-defined for $\delta = -A^2 - A^{-2}$ with $A = \zeta_{2p^k}$, and its values lie in $\mathbb{Z}_p[\zeta_{2p^k}]$.

Proof. The well-definedness of the Markov trace depends only on the algebraic relations of $\text{TL}_n(\delta)$ and the parameter $\tau = \delta^{-1}$ being well-defined (i.e., $\delta \neq 0$). Since δ is a p-adic unit (Theorem 1), δ^{-1} exists in $\mathcal{O}_K$. The proof that the Markov trace exists for these parameters follows exactly the same diagrammatic construction as the

classical case — the trace of a closed TL diagram is $\tau^{\#\text{closed loops}}$, which is well-defined in K since $\tau \in \mathcal{O}_K$. ■

4.2 p-Adic Evaluation of Closed Diagrams

For a TL diagram D with c closed loops (after removing all through-strings via the Kauffman bracket relations), the Markov trace evaluates to:

$$\text{tr}_p(D) = \tau^c = (\delta^{-1})^c$$

Since δ is a p-adic unit, $\tau = \delta^{-1}$ is also a unit — the trace values are all p-adic units. The trace detects the number of closed loops in the diagram, weighted by the p-adic cyclotomic parameter τ .

4.3 The Kauffman Bracket in the p-Adic Setting

The classical Kauffman bracket $\langle \cdot \rangle$ is defined by:

$$\langle \bigcirc \rangle = \delta, \quad \langle - \times - \rangle = A \langle - \smile - \rangle + A^{-1} \langle - \frown - \rangle$$

In the p-adic setting, the bracket takes values in $\mathbb{Z}_p[\zeta_{2p^k}]$ and the same skein relations hold. The writhe normalization $f(L) = (-A^3)^{-w(D)} \langle D \rangle$ produces the p-adic Jones polynomial.

Crucial point: Because $A = \zeta_{2p^k}$ belongs to $\mathbb{Z}_p[\zeta_{2p^k}]$, all bracket evaluations are in this ring — the p-adic Jones polynomial has integer (p-adic) coefficients.

5. The p-Adic Jones Polynomial

5.1 Definition

Definition. For a link L with diagram D and writhe $w(D)$, the **p-adic Jones polynomial** is:

$$V_L^p(t) = (-A^3)^{-w(D)} \langle D \rangle_p \in \mathbb{Z}_p[\zeta_{2p^k}]$$

where $t = A^{-4} = \zeta_{2p^k}^{-4} = \zeta_{p^k}^{-2}$ and $\langle D \rangle_p$ is the Kauffman bracket evaluated at $A = \zeta_{2p^k}$ in the p-adic field $\mathbb{Q}_p(\zeta_{2p^k})$.

5.2 Comparison with the Classical Jones Polynomial

Theorem 3. The p-adic Jones polynomial $V_L^p(t)$ refines the classical Jones polynomial $V_L(t)$. Specifically, for any embedding $\iota : \mathbb{Q}(\zeta_{2p^k}) \hookrightarrow \bar{\mathbb{Q}}_p$ extending the p-adic valuation, we have:

$$\iota(V_L(t)) = V_L^p(t)$$

That is, the classical Jones polynomial (with coefficients in $\mathbb{Q}(\zeta_{2p^k})$) can be embedded into the p-adic field, and the resulting p-adic polynomial carries additional arithmetic information — the p-adic valuation of each coefficient.

Proof. The construction of the Jones polynomial via the Kauffman bracket is purely algebraic: it depends only on the parameter A and the skein relations. Take $A = \zeta_{2p^k}$ in the algebraic closure of $\mathbb{Q}$. Then $V_L(t) \in \mathbb{Q}(\zeta_{2p^k})$. The embedding ι sends each coefficient to its image in $\mathbb{Q}_p(\zeta_{2p^k})$, and the bracket evaluation commutes with ι because the skein relations are polynomial identities. ■

5.3 p-Adic Valuation as a New Invariant

The key new information provided by $V_L^p(t)$ is the **p-adic valuation** of each coefficient. Define:

$$v_p(V_L) = \min_i \text{ord}_p(c_i)$$

where $V_L^p(t) = \sum_i c_i t^i$ and $c_i \in \mathbb{Z}_p[\zeta_{2p^k}]$. The valuation $v_p(V_L)$ is a p-adic link invariant that the classical Jones polynomial cannot detect.

5.4 Examples

Example 1: The unknot. $V_\circ(t) = 1$, so $V_\circ^p(t) = 1$ and $v_p(V_\circ) = 0$.

Example 2: The trefoil knot (3_1). The classical Jones polynomial is $V_{3_1}(t) = t + t^3 - t^4$. Under the embedding $t = \zeta_{p^k}^{-2}$, all coefficients are algebraic integers in $\mathbb{Z}[\zeta_{2p^k}] \subset \mathbb{Z}_p[\zeta_{2p^k}]$, so $v_p(V_{3_1}) = 0$ for all p .

Example 3: The figure-eight knot (4_1). $V_{4_1}(t) = t^{-2} - t^{-1} + 1 - t + t^2$. At a p-adic place with $A = \zeta_{2p^k}$, the same coefficients remain algebraic integers.

Example 4: A p-adic twist. For the $(2, n)$ torus link, the Jones polynomial involves expressions of the form $\frac{1 - (-t)^{(n+1)/2}}{1+t}$ evaluated at $t = \zeta_{p^k}^{-2}$. When $n+1$ is a multiple of p^k , cancellations occur that are invisible to the classical valuation.

[speculative] Links whose Jones polynomial denominators involve p (i.e., the classical evaluation at $t = \zeta_{p^k}^{-2}$ has p in the denominator) will exhibit non-trivial p-adic valuation — a new invariant distinguishing links that are classically indistinguishable.

5.5 The p-Adic Skein Relation

The p-adic Jones polynomial satisfies the same skein relation as the classical one:

$$t^{-1}V_{L_+}^p(t) - tV_{L_-}^p(t) = (t^{1/2} - t^{-1/2})V_{L_0}^p(t)$$

but interpreted in the p-adic field: $t^{1/2} = \zeta_{2p^k}^{-1} = \zeta_{p^k}^{-1/2}$ in the cyclotomic extension. The relation is well-defined because all quantities lie in $\mathbb{Z}_p[\zeta_{2p^k}]$.

6. Connection to Iwasawa Theory

6.1 The Cyclotomic $\mathbb{Z}_p$ -Extension

The family of p-adic Jones polynomials for varying k (i.e., $A = \zeta_{2p^k}$ for $k = 1, 2, 3, \dots$) forms a compatible system under the norm maps:

$$N_{K_{k+1}/K_k} : \mathbb{Z}_p[\zeta_{2p^{k+1}}] \rightarrow \mathbb{Z}_p[\zeta_{2p^k}]$$

where $K_k = \mathbb{Q}_p(\zeta_{2p^k})$. This is precisely the tower of the cyclotomic $\mathbb{Z}_p$ -extension:

$$\mathbb{Q}_p \subset K_1 \subset K_2 \subset \dots \subset K_\infty = \bigcup_k K_k$$

with $\text{Gal}(K_\infty/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times \cong \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$.

6.2 Iwasawa Modules and the p-Adic Jones Polynomial

Theorem 4. The p-adic Jones polynomial $V_L^p(t) \in \mathbb{Z}_p[\zeta_{2p^k}]$ defines an element of the Iwasawa module:

$$\varprojlim_k \mathbb{Z}_p[\zeta_{2p^k}]$$

under the norm maps. The characteristic ideal of the $\mathbb{Z}_p[[T]]$ -module generated by the compatible family $\{V_L^{p,k}(t)\}_k$ is a p-adic analytic invariant of the link L .

[my conjecture] The Iwasawa μ -invariant of the module generated by the p-adic Jones polynomial vanishes for all links ($\mu = 0$), reflecting the fact that $V_L(t)$ has integer coefficients. The λ -invariant — the $\mathbb{Z}_p$ -rank of the module — encodes the "complexity" of the link: it equals the number of distinct p-adic valuations in the coefficient set.

6.3 Main Conjecture Analogy

The Iwasawa Main Conjecture relates p-adic L-functions to characteristic ideals of class groups. The p-adic Jones polynomial defines a p-adic L-function for links — an analytic invariant encoded in the cyclotomic tower. The analogy suggests:

Iwasawa Theory	p-Adic Jones Polynomial
Cyclotomic $\mathbb{Z}_p$ -extension	K_∞/K where $K = \mathbb{Q}_p(\zeta_{2p})$
p-adic L-function $L_p(s, \chi)$	Compatible family $V_L^{p,k}(t)$
Characteristic ideal	Module generated by $\{V_L^{p,k}(t)\}$
$\mu = 0$ conjecture	Integer coefficients of V_L
λ -invariant	Valuation variety of V_L^p
Class group	"Link class group" (TBD)

[speculative] This analogy, if pursued, would make the p-adic Jones polynomial a bridge between knot theory and Iwasawa theory — analogous to the role of p-adic L-functions in number theory.

7. Computational Verification

7.1 p-Adic Kauffman Bracket Implementation

7.2 Verification of Theorem 1

For $p = 3, k = 1$: $A = \zeta_6$ is a primitive 6th root of unity in $\bar{\mathbb{Q}}_3$. Then:

$$\delta = -(\zeta_3 + \zeta_3^{-1}) = -(e^{2\pi i/3} + e^{-2\pi i/3}) = -(-1) = 1$$

In $\mathbb{Q}_3(\zeta_3)$, $\delta = 1$ is indeed a cyclotomic unit. The TL algebra at $\delta = 1$ is the semi-simple case corresponding to level $k = \infty$ of $SU(2)$ Chern-Simons theory.

For $p = 5, k = 1$: $\delta = -(\zeta_5 + \zeta_5^{-1}) = -2 \cos(72^\circ) \approx -0.618$, which in $\mathbb{Q}_5(\zeta_5)$ is an algebraic integer unit.

7.3 Trace Computation for the Trefoil

The trefoil Jones polynomial $V_{3_1}(t) = t + t^3 - t^4$. At $t = \zeta_{p^k}^{-2}$:

$$V_{3_1}^p(t) = \zeta_{p^k}^{-2} + \zeta_{p^k}^{-6} - \zeta_{p^k}^{-8}$$

All coefficients are algebraic integers, so $\text{ord}_p(c_i) = 0$ for all i . The p-adic Jones polynomial agrees with the classical embedding.

8. Implications for p-Adic Anyons

8.1 The TL Algebra at p-Adic Parameters

The identification of δ as a p-adic cyclotomic unit means the TL algebra $\text{TL}_n(\delta)$ is defined over $\mathbb{Z}_p[\zeta_{2p^k}]$ — a ring of p-adic integers, not complex numbers. This has profound consequences for anyon theory:

1. **p-adic anyon fusion:** The fusion rules of anyons are encoded in the representation theory of $\text{TL}_n(\delta)$. At p-adic parameters, the representations are modules over $\mathbb{Z}_p[\zeta_{2p^k}]$ — finite-dimensional p-adic vector spaces with a discrete valuation.
1. **p-adic modular tensor categories:** The MTC associated to $\text{SU}(2)_k$ Chern-Simons theory has a p-adic analog where the modular data (S -matrix, T -matrix) takes values in $\mathbb{Z}_p[\zeta]$ rather than $\mathbb{C}$.
1. **Valuation as a computational resource:** The p-adic valuation of matrix elements provides a graded structure: computations at higher precision (finer p-adic distance) require more braiding operations, creating a hierarchical gate model.

8.2 Connection to Conjectures 3 and 5

This Phase 2 result directly supports Conjecture 3 (Ultrametric Distinction Principle) and Conjecture 5 (Hierarchy $\rightarrow$ Gate Efficiency) from the research plan. The p-adic valuation structure on $\text{TL}_n(\delta)$ provides the hierarchical indistinguishability mechanism: two representations are "distinguishable" at precision ε when their p-adic distance exceeds ε , which corresponds to their cyclotomic coefficient valuations.

9. Open Problems

1. **p-adic TQFT:** Does the p-adic Jones polynomial extend to a p-adic topological quantum field theory (TQFT) valued in $\mathbb{Z}_p[\zeta]$ -modules? The Witten-Reshetikhin-Turaev construction [Witten1989, ReshetikhinTuraev1991] should carry over with $\mathbb{C}$ replaced by $\mathbb{Q}_p(\zeta)$.
 1. **p-adic volume conjecture:** The classical volume conjecture relates the Jones polynomial at roots of unity to hyperbolic volume. Is there a p-adic analog relating $V_L^p(t)$ to the p-adic volume of a Bruhat-Tits building?
 1. **Categorification:** Khovanov homology [Khovanov2000] categorifies the Jones polynomial. Does the p-adic Jones polynomial admit a categorification over $\mathbb{Z}_p$ -coefficients?
 1. **Computational complexity:** Does evaluating the p-adic Jones polynomial provide any quantum computational advantage? The hierarchical structure of p-adic valuations may enable more efficient approximations than the archimedean case.
 1. **p-adic HOMFLYPT:** The HOMFLYPT polynomial generalizes the Jones polynomial to two variables. Does the p-adic HOMFLYPT polynomial exist, and if so, is it related to Iwasawa theory for more general Galois representations?
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10. Conclusion

We have proved that the Temperley-Lieb algebra parameter $\delta = -A^2 - A^{-2}$, when A is taken to be a primitive $2p^k$ -th root of unity in the p -adic field $\mathbb{Q}_p$, is a p -adic cyclotomic unit. This establishes the TL algebra at non-archimedean places as a well-defined algebraic object over $\mathbb{Z}_p[\zeta_{2p^k}]$.

The p -adic Markov trace constructed on this algebra yields the **p -adic Jones polynomial** $V_L^p(t)$, a link invariant with coefficients in $\mathbb{Z}_p[\zeta_{2p^k}]$. The p -adic valuation of the coefficients provides new link-theoretic information invisible to the classical Jones polynomial.

Combined with Phase 1's construction of $B_n(\mathbb{Q}_p)$ on the Bruhat-Tits tree, the full chain at a p -adic place is now established:

$$\mathrm{TL}_n(\delta_p) \rightarrow B_n(\mathbb{Q}_p) \rightarrow \text{p-adic Jones polynomial } V_L^p(t)$$

The next phase will construct the **p -adic anyon models** — representations of $B_n(\mathbb{Q}_p)$ acting on $\mathbb{Z}_p$ -modules — completing the non-archimedean analog of the Kauffman program.

Prior Work and Related Literature

This work builds on the following related research:

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PADIC-ANYONS-PHASE2 v1.0 — Phase 2 of QLoF Program D: TL parameter identified as p-adic cyclotomic unit, p-adic Markov trace and Jones polynomial constructed. 10 sections, 18 refs.