

Projective Braid Group Representations from a Microscopic Long-Range Kitaev Chain

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Abstract

We construct an explicit projective representation of the braid group from the microscopic Hilbert space of a Kitaev chain extended with algebraically decaying long-range hopping and pairing. The construction proceeds in three steps. First, we identify the topological phase of the long-range Kitaev chain (power-law exponent α) and show that unpaired Majorana zero modes (MZMs) localize at domain walls, with localization length controlled by α . Second, we write the braid unitary $U_{ij} = \exp((\pi/4)\gamma_i\gamma_j)$ acting on the zero-mode Majorana operators γ_i and verify, by explicit matrix computation in a fixed-parity sector, that the braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ holds exactly while the relation $\sigma^4 = 1$ is lifted: $U^4 = -\mathbb{1}$ rather than $+\mathbb{1}$. This defines a genuine projective (2-cocycle-twisted) representation whose class is the Ising one, with per-exchange projective phase anchored at $\pi/8$. Third, we give quantitative estimates of the physical requirements: zero-mode splitting, gap, and cumulative phase drift under long-range perturbations. Our central quantitative result is that the Ising projective class is stable against long-range hopping with exponent $\alpha > 2$ and open gap: our projection at $\alpha = 3$ with representative parameters — based on the assumed, undischarged scaling $\varepsilon \sim \lambda s \cdot s^{-\alpha}$ rather than a derived bound — gives a per-braid drift of $\leq 1.5 \times 10^{-17}$ rad and a cumulative drift over 10^6 braids of $\leq 9.2 \times 10^{-11}$ rad. We connect the construction to projective representations in Chern–Simons theory, to projective ribbon permutation statistics in three dimensions, and to recent results on the non-uniqueness of modular data from braid representations. Falsification criteria and open questions are discussed.

1. Introduction

Topological phases of matter support exotic quantum statistics: when quasiparticles are exchanged adiabatically, the wave function transforms by a unitary matrix, and for non-Abelian anyons these matrices form a representation of the braid group B_N . A recurring theme across exactly soluble field theories is that such representations are generically *projective*: the unitaries satisfy the group relations only up to phases, because the phase ambiguity is the fingerprint of a nontrivial central extension or 2-cocycle. Canonical Chern–Simons theory on a Riemann surface provides the canonical example: its wave functions carry projective representations whose cocycles depend on the (possibly rational) Chern–Simons level [1, 2], and the symplectic approach to the Wess–Zumino–Witten model yields projective representations of the loop group by the same mechanism [4].

The purpose of this paper is to derive such a projective braid representation not from a topological field theory but from a *microscopic lattice model*: the Kitaev chain, generalized to include long-range (power-law decaying) hopping and pairing. The motivation is threefold.

First, microscopic derivations anchor the abstract theory. Chern–Simons derivations [1, 2] are continuum and topological; a lattice derivation shows precisely which microscopic operators generate the braid action and where the projective phase comes from — it comes from the Clifford algebra $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$, i.e., from fermionic anticommutation, not from any gauge field.

Second, long-range hopping changes the locality structure. In the short-range Kitaev chain the MZMs are exponentially localized; with hopping $t_\ell \sim t_0/\ell^\alpha$, localization and splitting scale differently, and for $\alpha \leq 1$ the zero modes can fail to be localized at all. We quantify this and show that the *algebraic* structure of the braid representation is independent of α , while its *physical* realizability is not.

Third, there is a live question about what braid representations do and do not determine. Recent analyses indicate that braid group representations of non-Abelian anyons need not uniquely fix the modular data (S and T matrices) of the underlying topological order, with the Ising theory as the canonical test case [12, 13]. A microscopic lattice construction of the braid representation, decoupled from any modular tensor category, is a useful control experiment for such uniqueness questions.

Our main results are: (i) an explicit verification that the Majorana braid matrices satisfy the braid relation exactly and the relation $\sigma^4 = 1$ only projectively, with $U^4 = -\mathbb{1}$ in the fixed-parity sector; (ii) explicit numerical estimates of zero-mode localization, gap, and cumulative phase drift for representative long-range parameters, showing class stability for $\alpha > 2$; and (iii) a roadmap for completing the derivation, including the marginal regime $\alpha \in (1, 2]$ where the standard short-range arguments require modification.

2. Background and Related Work

Chern–Simons braid representations. The canonical route to anyon statistics is canonical quantization of Chern–Simons theory coupled to non-dynamical matter. In [1], the quantization is generalized to rational Chern–Simons coefficient k , arbitrary genus, and arbitrary total charge, and the question is posed of which braid group representations on a Riemann surface the wave functions carry. In [2], an explicit solution of the Schrödinger equation for Chern–Simons theory coupled to charged particles is given for rational k and zero total charge, and the wave functions are shown to carry a *projective* representation of the group of large gauge transformations. These works establish the template our lattice construction mirrors: a quantization ambiguity (rational k there, fermion parity here) produces a projective rather than honest representation.

Symplectic and loop-group perspectives. The symplectic approach to the Wess–Zumino–Witten model [4] obtains the quantum theory as a projective representation of the loop group, with Gauss constraints imposed at a finite set of points. This is conceptually parallel to our construction: the Majorana zero modes are precisely "Gauss constraints at points" (defects), and the projectivity arises from quantizing a symplectic space whose prequantum line bundle has nonzero curvature. The Kashaev quantization of universal Teichmüller space similarly yields projective representations of the Ptolemy–Thompson group and associated dilogarithmic central extensions [5]; the structural lesson — that quantization of a classical symmetry group naturally produces central extensions — is the same one we instantiate for B_N .

Projective statistics beyond braiding. The analysis in [3] of non-Abelian braid statistics versus projective permutation statistics clarifies that non-Abelian projective representations of the permutation group can serve as statistics valid in any dimension, but that these are strictly different structures from braid statistics. This distinction matters for us: a strictly 1D chain can at best realize the projective-permutation sector unless the chain is embedded in a network that permits genuine exchange. In a related direction, [8] analyzes the three-dimensional model of Teo and Kane, in which defect lines carry Majorana zero modes whose exchange and twist implement a "ghostly" remnant of 2D Ising-anyon braiding; the governing group T_{2n} (projective ribbon permutation group) is computed there. Our work is the 1D-chain analogue: the Kitaev chain's MZMs, moved around a network, implement braid-like unitaries whose projectivity is the same remnant phenomenon.

Algebraic machinery. The Drinfeld associator machinery [6] associates to representations of a certain Hopf algebra braid group representations over a Laurent-series field and studies their dependence on the associator for GT-rigid representations; this rigidity is the algebraic shadow of our physical claim that the cohomology class of the statistics cocycle cannot change without a phase transition. The cluster-geometric construction of [7] endows moduli spaces of G-local systems on decorated surfaces with a cluster Poisson structure equivariant under the mapping class group, providing a geometric arena in which braid-type actions arise from quantization of cluster varieties. Braid group actions on representation categories also appear in [11], where an action of the braid group of a simple Lie algebra on imaginary roots of a quantum affine algebra determines cyclicity of tensor products and zeros of R-matrices; and in [9], where brane quantization connects Lagrangian branes on the $SL(2, \mathbb{C})$ character variety to representations of the spherical double affine Hecke algebra of $C^\vee C_1$, with an affine braid group action on the category. Both illustrate that braid actions on zero-mode/defect Hilbert spaces are a robust algebraic phenomenon.

Modular data and uniqueness. Two recent analyses [12, 13] address whether the braid group representation carried by N non-Abelian anyons uniquely determines the modular data of the underlying topological order, with the Ising/Majorana fusion rules as the central test case; the emerging picture is one of non-uniqueness. Our microscopic construction is directly relevant: since the Majorana braid representation we derive in Section 4 is fixed entirely by the Clifford algebra, any "modular data" attributed to it must come from extra structure, not from the braid representation alone. Finally, [14] studies ultrametric relaxation dynamics in topological quantum memory, which bears on the stability of any memory built from our zero modes: the hierarchical relaxation barriers are a plausible model for error processes in a Majorana-based qubit whose protection is weakened by long-range hopping.

We note that reference [10] (a QCD collider-physics report) is part of the provided bibliography but is topically unrelated to this work and is not cited substantively.

3. Methods

3.1 The long-range Kitaev chain

The standard Kitaev chain on L sites is $H_0 = -\sum_j [\mu a^\dagger_j a_j + t(a^\dagger_j a_{j+1} + \text{h.c.}) + \Delta(a_j a_{j+1} + \text{h.c.})]$, with t, Δ real. We generalize to long-range hopping and pairing with algebraic decay: $H_\alpha = -\sum_j \sum_{\ell=1}^{L-j} [(t/\ell^\alpha)(a^\dagger_j a_{j+\ell} + \text{h.c.}) + (\Delta/\ell^\alpha)(a_j a_{j+\ell} + \text{h.c.})] - \mu \sum_j a^\dagger_j a_j$, with power-law exponent $\alpha > 0$. The model is in symmetry class BDI (or D with a phase on Δ); in the topological regime $|\mu| < 2t \sum_{\ell=1}^{\infty} \ell^{-\alpha} = 2t \zeta(\alpha)$ (valid for $\alpha > 1$, where $\sum_{\ell=1}^{\infty} \ell^{-\alpha} = \zeta(\alpha)$), the open chain hosts unpaired MZMs γ_1, γ_2 at its ends, with $\gamma_1 = \sum_j v_\alpha(j)(a_j + a^\dagger_j)$, $\gamma_2 = \sum_j (-1)^j v_\alpha(j)(a_j + a^\dagger_j)$, where v_α is a normalized Majorana wave function whose decay is controlled by α . For $\alpha > 1$ the decay is exponential with a length $\xi(\alpha)$ that grows as $\alpha \rightarrow 1^+$; for $\alpha \leq 1$ the zero modes delocalize and topological protection is lost. We treat $\xi(\alpha)$ as a quantity to be bounded, not assumed.

3.2 From zero modes to a braid action

Given $2n$ well-separated MZMs $\gamma_1, \dots, \gamma_{2n}$ with $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$, an adiabatic exchange of modes i and j implements, on the zero-mode Hilbert space, the unitary $U_{ij} = \exp((\pi/4) \gamma_i \gamma_j)$, whose action on the Majorana operators we verify directly in Section 4. The generators $\sigma_i := U_{\{i, i+1\}}$ are the candidate braid generators. The Hilbert space of $2n$ Majoranas is the Fock space of n complex fermions, with 2^n states; fixing total parity (superselection) gives a representation of dimension 2^{n-1} .

3.3 Exchange protocol in a chain

In a strictly 1D chain, "exchange" is not a native motion; we assume the standard network implementation: the chain is patterned into a T-junction network in which Majoranas are moved by tuning local chemical potentials, or exchanges are implemented by parity measurements. The braid unitary is the same either way; only the adiabaticity condition changes.

3.4 Cocycle extraction and stability criterion

For a loop ℓ in the ground-state manifold representing a braid σ_i , the geometric phase $\varphi(\ell)$ defines a normalized 2-cocycle $c(\sigma_i, \sigma_j) = \varphi(\sigma_i \sigma_j) - \varphi(\sigma_i) - \varphi(\sigma_j) \bmod 2\pi$ on BN with values in $U(1)$; its class in $H^2(BN, U(1)) \cong \mathbb{Z}_2$ (for $N \geq 5$, since $H_2(BN; \mathbb{Z}) \cong \mathbb{Z}_2$ and $H^1(BN) = 0$, so the universal-coefficient theorem gives $H^2(BN; U(1)) \cong \text{Hom}(H_2(BN), U(1)) \cong \mathbb{Z}_2$) is the invariant of interest. The Ising statistics corresponds to the nontrivial element of this $\mathbb{Z}_2$; the projective phase $e^{i\pi/8}$ is a representative-level quantity, with the class fixed only by the parity of the cocycle. The *stability criterion* is: the class is invariant under deformation of the Hamiltonian provided the bulk gap never closes along the path. For the long-range chain, the couplings t/ℓ^α are perturbatively irrelevant for the zero-mode sector when $\alpha > 2$ (the analogue of the familiar locality threshold $\alpha > d + 1$ in dimension $d = 1$), so the gap stays open and the class is protected; only the cocycle *representative* can drift, by an amount controlled by the zero-mode overlap.

4. Analysis

All numbers in this section are computed explicitly from stated inputs.

4.1 The braid unitary and its action on Majoranas

Let $A = \gamma_i \gamma_j$ with $i \neq j$. Then $A^2 = \gamma_i \gamma_j \gamma_i \gamma_j = -\gamma_i^2 \gamma_j^2 = -1$, so A has eigenvalues $\pm i$, and $U_{ij} = e^{i(\pi/4)A} = \cos(\pi/4) \mathbb{I} + \sin(\pi/4) A = (1/\sqrt{2})(\mathbb{I} + \gamma_i \gamma_j)$. Using the adjoint action: $[A, \gamma_i] = A\gamma_i - \gamma_i A = -\gamma_j - \gamma_j = -2\gamma_j$ and $[A, \gamma_j] = \gamma_i - (-\gamma_i) = 2\gamma_i$. The series $e^{\theta \text{ad} A} \gamma_i = \gamma_i \cos 2\theta - \gamma_j \sin 2\theta$ then gives, at $\theta = \pi/4$: $U_{ij} \gamma_i U_{ij}^\dagger = -\gamma_j$, $U_{ij} \gamma_j U_{ij}^\dagger = \gamma_i$. The exchange acts as $i \leftrightarrow j$ up to a sign convention (either sign is a valid orientation convention for the braid generator).

4.2 Explicit matrix computation: braid relation and projective lift

Take three Majoranas $\gamma_1, \gamma_2, \gamma_3$; fix the parity sector spanned by $\{|0\rangle, \gamma_3|0\rangle\}$, where $|0\rangle$ is the vacuum of the complex fermion $f = (\gamma_1 + i\gamma_2)/2$. In this basis, choosing phases consistently, $\gamma_1\gamma_2 = i\sigma_3$ and $\gamma_2\gamma_3 = i\sigma_1$. With $\theta = \pi/4$: $U_{12} = e^{i(\pi/4)(i\sigma_3)} = \text{diag}(e^{i\pi/4}, e^{-i\pi/4})$, $U_{23} = e^{i(\pi/4)(i\sigma_1)} = (1/\sqrt{2})[[1, i], [i, 1]]$.

Fourth power (projectivity). $U_{12}^4 = e^{i\pi\sigma_3} = \text{diag}(e^{i\pi}, e^{-i\pi}) = -\mathbb{I}$. So the braid relation $\sigma^4 = 1$ is satisfied only up to the phase -1 : the representation is projective. Equivalently, $U_{12}^2 = i\sigma_3 = i\gamma_1\gamma_2$: the square of an exchange is the fermion parity operator, and four exchanges differ from the identity by the fermion parity $(-1)^F$ — the well-known "fermion parity anomaly" of Majorana braids, here derived rather than assumed.

Braid relation, exact. Direct 2×2 multiplication, first ordering: $U_{12}U_{23} = (1/\sqrt{2})[[e^{i\pi/4}, i e^{i\pi/4}], [e^{-i\pi/4}, e^{-i\pi/4}]]$; right-multiplication by the diagonal U_{12} rescales the columns by $e^{i\pi/4}$ and $e^{-i\pi/4}$, giving $U_{12}U_{23}U_{12} = (1/\sqrt{2})[[e^{i\pi/2}, i], [i, e^{-i\pi/2}]] = (1/\sqrt{2})[[i, i], [i, -i]] = (i/\sqrt{2})[[1, 1], [1, -1]] = (i/\sqrt{2})(\sigma_1 + \sigma_3)$. Second ordering: $U_{23}U_{12} = (1/\sqrt{2})[[e^{i\pi/4}, i e^{-i\pi/4}], [i e^{i\pi/4}, e^{-i\pi/4}]]$; left-multiplication by U_{23} gives entrywise $(1,1) = (1/2)(e^{i\pi/4} - e^{-i\pi/4}) = i/\sqrt{2}$, $(1,2) = (i/2)(e^{i\pi/4} + e^{-i\pi/4}) = i/\sqrt{2}$, $(2,1) = i/\sqrt{2}$, $(2,2) = (1/2)(e^{-i\pi/4} - e^{i\pi/4}) = -i/\sqrt{2}$, so $U_{23}U_{12}U_{23} = (i/\sqrt{2})[[1, 1], [1, -1]] = (i/\sqrt{2})(\sigma_1 + \sigma_3)$. Both orderings therefore agree exactly, $U_{12}U_{23}U_{12} = U_{23}U_{12}U_{23} = (i/\sqrt{2})[[1, 1], [1, -1]]$, using the same orientation convention

($\gamma_1\gamma_2 = i\sigma_3$, $\gamma_2\gamma_3 = i\sigma_1$, forcing $\gamma_1\gamma_3 = -i\sigma_2$) for both. The braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ therefore holds *exactly* (not merely projectively) in this representation, while $\sigma^4 = 1$ holds only projectively. This is precisely the Ising anyon structure: the exchange phases in the two fusion channels are $R^{\{\sigma\sigma\}}_1 = e^{\{-i\pi/8\}}$ and $R^{\{\sigma\sigma\}}_{-\psi} = e^{\{3i\pi/8\}}$ (consistent with the Ising test case of [12]), and the per-exchange projective phase relative to a bosonic baseline is $\varphi_{\text{Ising}} = \pi/8 = 3.14159/8 = 0.3927$ rad.

4.3 Input numbers

Symbol	Description	Value	Source
α	Power-law exponent	3 (representative)	chosen
μ/t_0	Chemical potential ratio	0.1	chosen
$\zeta(3)$	Riemann zeta	1.2020569	standard
s	MZM separation	10 sites	chosen
φ_{Ising}	Short-range anchor phase	$\pi/8 = 0.3927$ rad	Sec. 4.2, [12]

4.4 Localization and drift bounds

Localization parameter. For a semi-infinite chain with $t(r) = \text{tor}^{\{-\alpha\}}$, the MZM decay factor λ satisfies, at small μ , $\lambda \approx \mu/(2t_0\zeta(\alpha))$. At $\alpha = 3$, $\mu/t_0 = 0.1$: $\lambda = 0.1/(2 \times 1.2020569) = 0.1/2.4041138 = 0.0416$. The localization length is $\xi = 1/|\ln \lambda| = 1/3.1792 = 0.3145$ sites — essentially on-site localized — so the overlap of MZMs separated by s sites is $\sim \lambda^s$.

Drift projection at $\alpha = 3$ (assumption-labeled). We *assume*, without derivation, that the long-range correction to the exchange phase scales as $\varepsilon \sim \lambda^s \cdot s^{\{-\alpha\}}$. No Berry-connection calculation or perturbative derivation of the overlap-to-phase coupling is given here, and no prefactor is fixed; this is a projection, not a proved bound, and is subject to the same assumption-auditing as the $\alpha = 2.1$ estimate below. For $s = 10$, $\alpha = 3$: $\lambda^{10} = e^{\{10 \ln 0.0416\}} = e^{\{-31.79\}} = 1.46 \times 10^{-14}$; $\varepsilon = 1.46 \times 10^{-14} \times 10^{-3} = 1.46 \times 10^{-17}$. The per-braid phase drift is bounded by $|\Delta\varphi| \leq 2\pi\varepsilon$; over $N_{\text{pairs}} = 10^6$ braids, $|\Delta\varphi| \leq 2\pi \times 1.46 \times 10^{-17} \times 10^6 = 9.2 \times 10^{-11}$ rad — negligible under the stated assumption. Applying the same factor-of-3 uncertainty treatment as for the $\alpha = 2.1$ projection, the cumulative drift lies in $[3.1, 27.6] \times 10^{-11}$ rad. The Ising projective class is expected to be stable in this regime, contingent on the assumed scaling $\varepsilon \sim \lambda^s \cdot s^{\{-\alpha\}}$.

Gap. At $\mu = 0$, $\alpha = 3$: $\Delta_{\text{gap}} = 2\Delta_0\zeta(3) = 2 \times 1.2020569 \Delta_0 = 2.4041 \Delta_0 > 0$, confirming an open gap along the deformation path from $\alpha = \infty$ to $\alpha = 3$ at these parameters.

Dangerous regime (projection, clearly labeled). The above bound degrades as $\alpha \rightarrow 2^+$ and $\mu \rightarrow \mu_c = 2t_0\zeta(\alpha)$, where ξ diverges. *Projection with stated assumptions:* at $\alpha = 2.1$, $\mu/\mu_c = 0.9$, take $\xi \approx 10$ sites (from $\xi \sim \xi_0/(1 - \mu/\mu_c)$ with $\xi_0 \sim 1$) and $\varepsilon \sim e^{\{-s/\xi\}} s^{\{-\alpha\}} = e^{\{-1\}} \times 10^{\{-2.1\}} = 0.368 \times 0.00794 = 2.92 \times 10^{-3}$. Per-braid drift $\approx 2\pi \times 2.92 \times 10^{-3} \approx 1.8 \times 10^{-2}$ rad; over 10^3 braids, cumulative drift ~ 18 rad, destroying gate fidelity. Assumptions: (a) exponential localization with the stated ξ ; (b) the Berry-curvature correction is first order in ε ; (c) no gap closing. Each could fail by factors of order unity; we quote the per-braid drift as 1.8×10^{-2} rad with a factor-of-3 uncertainty band, $[0.6, 5.4] \times 10^{-2}$ rad.

4.5 Cohomological conclusion

In the short-range limit the cocycle representative takes values generated by $e^{\{i\pi/8\}}$; the Ising statistics corresponds to the nontrivial element of $H^2(\text{BN}, U(1)) \cong \mathbb{Z}_2$ ($N \geq 5$). *The drift computed above changes the representative but not the class, provided the deformation path never closes the gap. Note that the phase $e^{\{i\pi/8\}}$ itself is a representative-level quantity: the cohomology class distinguishes only the parity of the cocycle, i.e., Ising*

versus trivial statistics. Since $\alpha > 2$ keeps the long-range couplings perturbatively irrelevant and the gap open ($\Delta_{\text{gap}} = 2.4041 \Delta_0$ at $\alpha = 3, \mu = 0$), the class is invariant — the microscopic counterpart of the GT-rigidity observed algebraically in [6].

5. Results

1. **Braid unitary:** $U_{ij} = \exp((\pi/4)\gamma_i\gamma_j)$ exchanges $\gamma_i \leftrightarrow \mp\gamma_j$; verified by explicit adjoint-action computation.
2. **Projectivity:** $U^2 = i\gamma_i\gamma_j$ (fermion parity); $U^4 = -\mathbb{I}$ in the fixed-parity sector; braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ holds exactly. Cocycle value $\omega(\sigma, \sigma, \sigma) = -1$.
3. **Anchor phase:** $\varphi_{\text{Ising}} = \pi/8 = 0.3927$ rad per exchange, from the Ising R-matrix values $e^{\{-i\pi/8\}}, e^{\{3i\pi/8\}}$.
4. **Representation dimension:** $2^{\{n-1\}}$ on $2n$ Majoranas at fixed parity.
5. **Localization at $\alpha = 3, \mu/\text{to} = 0.1$:** $\lambda = 0.0416, \xi = 0.3145$ sites.
6. **Drift projection at $\alpha = 3, s = 10$** (assumed scaling $\varepsilon \sim \lambda^s \cdot s^{\{-\alpha\}}$, not derived): $\varepsilon = 1.46 \times 10^{-17}$ per braid; cumulative over 10^6 braids $\leq 9.2 \times 10^{-11}$ rad, with factor-of-3 band $[3.1, 27.6] \times 10^{-11}$ rad. The nontrivial $\mathbb{Z}_2$ projective class (Ising) is expected to be stable under these assumptions.
7. **Gap at $\mu = 0, \alpha = 3$:** $\Delta_{\text{gap}} = 2.4041 \Delta_0$.
8. **Projection ($\alpha = 2.1, \mu/\mu_c = 0.9$):** per-braid drift $\approx 1.8 \times 10^{-2}$ rad, band $[0.6, 5.4] \times 10^{-2}$ rad; ~ 18 rad over 10^3 braids — loss of topological protection near the marginal regime.

6. Discussion

Limitations. First, the drift bound in the localized regime relies on the self-consistency equation for λ , which is exact only for the pure power-law chain at small μ ; realistic devices have chemical-potential disorder that can resonantly enhance zero-mode overlaps by orders of magnitude, invalidating the 10^{-17} estimate. Second, the cohomological invariance argument assumes the deformation path never closes the bulk gap; long-range chains are known to exhibit re-entrant gap closings at intermediate α , and a single gap closing would allow the class to jump between the two elements of $\mathbb{Z}_2$. Third, the analytic skeleton presented here should be checked by exact diagonalization (the model is quadratic, so exact) for finite chains, $2n \leq 10$, before the marginal regime $\alpha \in (1, 2]$ is trusted. Fourth, the drift bound may miss nonperturbative accumulation from the nonlocal Majorana–Majorana coupling $H_{\text{eff}} \sim i\delta\gamma_1\gamma_2$ with $\delta \propto s^{\{-\alpha\}}$, which is only algebraically small just above $\alpha = 2$.

Failure modes and falsifiability. The central claim — class stability for $\alpha > 2$ — would be falsified by: (i) an exact-diagonalization computation of the exchange unitary for a finite long-range chain showing a phase outside $\{e^{\{-i\pi/8\}}, e^{\{3i\pi/8\}}\}$ up to smooth drift, at parameters where the gap is provably open; (ii) a demonstration that the long-range chain realizes a *different* central extension, e.g., the dilogarithmic extension of [5]; (iii) a demonstration that the relevant group is not BN *at all but the projective permutation group of [3]* — a *serious possibility for a strictly 1D chain, in which case the 3D remnant group $T\{2n\}$ of [8] is the correct target if the chain is embedded in a higher-dimensional defect network.*

Non-uniqueness warnings. The analysis [12] shows that even a complete braid representation does not uniquely fix modular data; hence our cocycle, even if derived perfectly, underdetermines any putative 2D topological order adjacent to the chain. The partial classification of MZM fusion rules in [13] means we cannot exclude exotic fusion channels beyond the Ising set from braid data alone. The relaxation analysis of [14] suggests that even correct statistics can be operationally swamped by ultrametric error dynamics in the encoded memory.

Open questions. (1) Does the long-range chain in the marginal regime $1 < \alpha \leq 2$ realize a continuous family of cocycle representatives, and can the class jump only via gap closings? (2) Can the brane–DAHA correspondence of [9] be made literal for the long-range chain? (3) What does the tensor-product cyclicity criterion of [11] imply about which fusion channels of N long-range-coupled MZMs are reachable? (4) How do disorder-induced overlap enhancements modify the drift bounds quantitatively?

7. Conclusion

We have constructed an explicit projective representation of the braid group from the microscopic Hilbert space of a long-range Kitaev chain. The projectivity originates entirely in the Clifford algebra of Majorana zero modes — the fermion parity anomaly $U^4 = -\mathbb{1}$ — and defines the nontrivial element of $H^2(B_N, U(1)) \cong \mathbb{Z}_2$ ($N \geq 5$), i.e., the Ising projective class. Long-range hopping with exponent $\alpha > 2$ leaves this class invariant, with projected numerically negligible drift of the cocycle representative ($\leq 9.2 \times 10^{-11}$ rad over 10^6 braids at $\alpha = 3$, under the assumed scaling $\varepsilon \sim \lambda^s \cdot s^{-\alpha}$), while the marginal regime $\alpha \rightarrow 2^+$ near the phase boundary destroys protection. The construction bridges microscopic lattice Hamiltonians and the abstract projective representations of Chern–Simons theory, WZW models, and higher-dimensional Majorana defect constructions, and provides a clean control experiment for questions of modular-data uniqueness. Future work should address exact finite-size verification, disorder effects, and the marginal regime.

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Appendix A. Divergence report

D1. Value and provenance of the exchange phase (C2).

- Draft A: $\varphi = (\pi/2)(t_2/t_1) = 0.15\pi \approx 0.471$ rad, obtained by *postulating* a linear interpolation $\varphi(r) = \varphi_0 r$ with $\varphi_0 = \pi/2$ and $r = t_2/t_1 = 0.3$ ($t_1 = 1$, $t_2 = 0.3$ attributed to the parameter regime of [8]). A itself flags this as a postulate,

not a derived result.

- Draft B: anchor phase $\varphi_{\text{Ising}} = \pi/8 = 0.3927$ rad, from the standard Ising R-matrix values $e^{\{-i\pi/8\}}, e^{\{3i\pi/8\}}$ (consistent with [12]).
- Draft C: generator convention $U = \exp((\pi/4)\gamma\gamma)$, giving $U^2 = \text{parity}$ and $U^4 = -\mathbb{1}$; the per-channel exchange phases are the Ising $e^{\{\pm i\pi/8\}}$ values.
- *Underlying disagreement*: A assumes the projective phase is a continuous function of the microscopic hopping ratio (linear interpolation convention); B and C assume the standard Ising/Clifford convention in which the phase is topologically quantized and cannot vary continuously without a phase transition. These conventions are incompatible: a continuous deformation of the phase at fixed class is impossible for the nontrivial isolated class in $H^2(B_N, U(1)) \cong \mathbb{Z}_2$ unless the representative (not the class) drifts.
- *Resolution adopted*: the main text uses the B/C convention (Ising anchor $\pi/8$, generator $\exp((\pi/4)\gamma\gamma)$), which is derived from the Clifford algebra rather than postulated. A's linear-interpolation proposal is not adopted; it is reported here as an unverified conjecture that would, if true, imply either a class change (requiring a gap closing) or a representative drift far larger than the bounds computed in Section 4.4.

D2. Effect of long-range hopping on the representation (C3).

- Draft A: long-range hopping "deforms the usual Ising anyon representation into a continuous family of projective representations" parameterized by t_2/t_1 .
- Drafts B, C: the algebraic structure of the braid representation is independent of α ; long-range hopping controls only localization, gap, and the drift of the cocycle representative; the class is stable for $\alpha > 2$ with open gap.
- *Underlying disagreement*: A treats the phase as a microscopic, continuously tunable quantity; B/C treat it as topologically protected. B/C's position is supported by the cohomological argument (class in $\mathbb{Z}_2$, invariant under gap-preserving deformations) and by explicit drift bounds.
- *Resolution adopted*: main text adopts the B/C position (class stability); A's continuous-family claim is documented here and listed as falsifiable (an exact-diagonalization demonstration of a continuously varying phase at open gap would overturn it).

D3. Magnitude of the short-range phase ($\pi/2$ vs $\pi/8$ vs $\pi/4$). A uses $\varphi_0 = \pi/2$; B uses $\pi/8$; C uses $\pi/4$ in the generator exponent. These are partially convention-dependent: $\pi/4$ is the generator angle ($U = \exp((\pi/4)\gamma\gamma)$), $\pi/2$ is the phase of U^2 relative to identity in a particular channel convention, and $\pi/8$ is the Ising topological spin per exchange relative to a bosonic baseline. The main text states all three relations explicitly (Section 4.2) so the conventions are unambiguous; no physical contradiction remains once the conventions are fixed.

Appendix B. Claim attribution

#	Claim	A	B	C	Status
C1	Kitaev chain hosts MZMs implementing Ising-type exchange statistics	Explicit MZM solutions (Sec. 3.1)	Clifford-algebra braid unitary (Sec. 4.1–4.2)	Same as B, with drift bounds (Sec. 4.4)	Adopted (B/C convention; anchor phase $\pi/8$)
C2	Value and provenance of the exchange phase	Postulated linear interpolation $\varphi = 0.15\pi$	Ising anchor $\pi/8$ from R-matrix	Generator $\exp((\pi/4)\gamma\gamma)$, channels $e^{\{\pm i\pi/8\}}$	Adopted (B/C); A's proposal reported as unverified conjecture (D1, D3)

#	Claim	A	B	C	Status
C3	Effect of long-range hopping on the representation	Continuous family of projective representations parameterized by t_2/t_1	Algebraic structure independent of α ; class stable for $\alpha > 2$	Same as B, with quantitative drift bounds	Adopted (B/C); A's continuous-family claim listed as falsifiable (D2)