

Spectral Decomposition of the Bruhat–Tits Tree as a Candidate Origin of Particle Mass Hierarchies: Exact Radial Spectrum, Finite Truncations, Semigroup Tests, and a Falsification Analysis

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Abstract

The Bruhat–Tits tree $\mathcal{T}_p$ of $\mathbb{Q}_p$ is the infinite $(p + 1)$ -regular tree that serves as the bulk of p -adic AdS/CFT. We test the conjecture that a natural Laplacian on $\mathcal{T}_p$ — or the prime-indexed arithmetic of the tree — encodes Standard Model mass ratios. We compute, with all arithmetic shown: (i) the exact adjacency spectrum $\sigma(A_p) = [-2\sqrt{p}, 2\sqrt{p}]$ and a proof that the ℓ^2 point spectrum of the adelic-conditioned Laplacian is empty for every prime p ; (ii) the finite-truncation spectrum for $p = 2$, $N = 3$, giving Laplacian eigenvalues $\mu_1 = 1$, $\mu_2 = 3$, $\mu_3 = 5$ and ratio $R = 5$, which we classify as a truncation artifact rather than a physical output; (iii) the Koide ratio $Q = 0.6666664$ for charged leptons, within 4×10^{-7} relative of $2/3$; and (iv) bounded semigroup tests of the $\mathbb{T}_{2,3,5} = \{2^a 3^b 5^c\}$ calibration hypothesis, which fail at the 3%–8% level with internal multiplicative inconsistency. Prime-indexed coincidences (e.g. $p = 211$ versus $m_\mu/m_e = 206.7683$, deviation 2.05%) carry no evidential weight given prime density. We conclude that the leading-order spectral encoding is falsified, specify what a surviving version would require, and provide a reproducible computational benchmark.

1. Introduction

The Bruhat–Tits tree $\mathcal{T}_p$ of the p -adic field $\mathbb{Q}_p$ is an infinite tree in which every vertex has degree $p + 1$; it plays, in p -adic geometry, the role that hyperbolic space plays in real geometry, and it is the bulk of the p -adic AdS/CFT correspondence [5]. A mature literature has transplanted relativistic and field-theoretic structures onto this tree: BTZ-type black-hole geometries and Wilson lines [2], boundary spinor theories obtained by integrating out the bulk [4], geodesic bulk diagrams computing conformal blocks [5], effective field theories on tree subspaces flowing to boundary conformal field theories [6], and formal verification of harmonic cochains in the Lean theorem prover [3]. What none of these works attempts — and what the present paper tests — is the stronger conjecture that the *spectrum* of a natural operator on $\mathcal{T}_p$ encodes the *mass spectrum* of particles, i.e. that number-theoretic geometry outputs phenomenological numbers.

The conjecture has an obvious appeal and an obvious danger. The appeal: the adelic product formula $\prod_v |x|_v = 1$ ties the tree at every prime p into a single rigid structure, so a spectral statement on $\mathcal{T}_p$ is not freely adjustable; if a mass ratio appeared as an eigenvalue ratio, its origin would be arithmetic, in the spirit of the adelic cross-domain programme [10], [12]. The danger: prime numbers are dense enough that almost any positive real number lies within a few percent of some prime or prime ratio, so agreement at the percent level is cheap and proves nothing without a mechanism fixing which prime, which eigenvalue, and which normalisation.

Our contribution is deliberately conservative and fourfold. First, we define the operator precisely (Section 3) and compute its exact radial spectrum by hand, including a proof that the ℓ^2 point spectrum is empty under the weakest adelic-compatible boundary conditions (Section 4). Second, we compute the spectrum of a finite Dirichlet truncation ($p = 2$, $N = 3$), a complementary model in which discrete eigenvalues do exist, and show that its characteristic ratio $R = 5$ is a truncation artifact. Third, we compute the Koide ratio for charged leptons and test the $\mathbb{T}_{2,3,5}$

semigroup calibration hypothesis against lepton and quark mass ratios with a bounded search. Fourth, we compare all resulting ratios against Standard Model mass ratios and argue that the leading-order encoding is falsified, while identifying the escape routes — non-radial boundary conditions, quotient trees, and dynamical (running) operators — that a surviving version would need.

2. Background and Related Work

We review the literature in three groups: tree geometry and arithmetic, p -adic holography, and spectral/adelic calibration programmes.

Tree geometry and arithmetic. Heyman [1] develops the theory of *branches* of the Bruhat–Tits tree: the set of maximal orders containing a given suborder in a matrix algebra over a local field forms a subtree, used for the global selectivity problem and local embedding computations. This matters for us because any boundary condition on $\mathcal{T}_p$ that privileges a vertex or ray is, in order-theoretic language, a choice of branch; [1] shows such choices are canonical arithmetic objects, not ad hoc graph-theoretic devices. Heyman [2] transplants the BTZ black hole onto $\mathcal{T}_p$, constructing a novel p -adic exponential function adapted to the tree and evaluating $\mathrm{PGL}(2, \mathbb{Q}_p)$ Wilson lines on the analogue BTZ connection. Their exponential function is the closest existing object to a radial eigenfunction of a tree Laplacian, and our radial spectrum can be read as the linearised version of their geometry. Heyman [3] machine-formalises the Bruhat–Tits tree in the Lean proof assistant and verifies a result on harmonic cochains — functions on the tree satisfying a discrete Laplace equation — establishing that the combinatorial facts we use (degree $p + 1$, no cycles, boundary $\mathbb{Q}_p$) are theorem-level, which raises the reproducibility bar for any spectral claim built on them. Heyman [8] defines p -adic colligations and transfer functions as maps from Bruhat–Tits trees to buildings, providing an operator-theoretic language (characteristic functions, conjugacy classes) in which spectral questions on the tree can be posed invariantly; this formalism could reorganise the spectral question in ways our radial reduction does not capture.

p -adic holography. Heyman [5] establishes that geodesic bulk diagrams on $\mathcal{T}_p$ compute global conformal blocks with boundary $\mathbb{Q}_p$ — the structural result that makes the tree a legitimate bulk spacetime with computable observables. Heyman [4] integrates out the interior of $\mathcal{T}_p$ for a bulk spinor field theory and finds a boundary theory resembling a free scalar CFT over $\mathbb{Q}_p$; their integration-out procedure is exactly the operation that would turn a bulk Laplacian spectrum into a boundary mass spectrum, so their result is the natural home for the encoding conjecture we test. Heyman [6] computes effective actions on two subspaces of $\mathcal{T}_p$ and shows both limits agree with the same p -adic boundary CFT — evidence that boundary data are insensitive to bulk details, which cuts against any claim that a specific bulk spectrum fixes observable numbers. Heyman [7] proves exponential mixing of the geodesic translation map on quotient trees $\Gamma \backslash \mathcal{T}$ under a non-arithmeticity condition on the length spectrum; the arithmetic/non-arithmetic dichotomy there is a warning for us: arithmetic spectra behave atypically, and mixing results do not transfer naively to arithmetic quotients.

Spectral and adelic calibration programmes. Heyman [9] documents spectral dynamics on Bruhat–Tits trees as a research record, and proposes that the Standard Model mass spectrum is organised by the Pythagorean semigroup $\mathbb{T}_{2,3,5} = \{2^a 3^b 5^c\}$ with a calibration register of fitted entries. Heyman [10] extends the programme to a full calibration register coupling the semigroup to tree geometry, including the claim that the diagonal embedding of $\mathbb{T}_{2,3,5}$ into the automorphism structure of $\mathcal{T}_p$ organises the mass hierarchy; Heyman [12] (the Phase 3–4 update) claims the semigroup is simultaneously the SM mass spectrum, a GKP code lattice, and an Efimov discretuum. Our exact radial spectrum and bounded semigroup tests provide the first clean falsification analysis of whether tree geometry alone can support such identifications. Heyman [11] (the Alpha Pi Project) explores fine-structure-constant

numerology in the same ecosystem; we treat it as context for the broader calibration programme rather than a direct target. None of [9]–[12] computes the point spectrum of the tree Laplacian with boundary conditions fixed by the adelic product formula; that gap is what Section 4 fills.

3. Methods

3.1 The tree and the operator

Let $\mathcal{T}_p$ be the Bruhat–Tits tree of $\mathbb{Q}_p$: the infinite $(p+1)$ -regular tree, every vertex of degree $q = p+1$, with no cycles, and boundary $\partial\mathcal{T}_p = \mathbb{Q}_p \cup \{\infty\}$ (facts established formally in [3]). The combinatorial Laplacian is

$$(\Delta f)(x) = \sum_{y \sim x} (f(x) - f(y)) = (p+1)f(x) - \sum_{y \sim x} f(y),$$

where $y \sim x$ denotes adjacency; equivalently $\Delta = (p+1)I - A$ with A the adjacency operator. Since constant shifts do not change eigenvalue ratios of positive eigenvalues, we work with A and translate at the end.

The ball of radius n around a fixed vertex v_0 has vertex count

$$B_p(n) = 1 + (p+1) \sum_{k=0}^{n-1} p^k = 1 + (p+1) \frac{p^n - 1}{p - 1},$$

since every vertex at distance $k \geq 1$ from v_0 has exactly one neighbour closer to v_0 and p neighbours farther.

3.2 Radial reduction

Fix a base vertex o . A function f is radial if $f(x)$ depends only on the graph distance $n = d(x, o)$. The radial subspace is invariant under A , on which A acts as the tridiagonal radial operator R :

$$(Rf)_n = f_{n-1} + p f_{n+1}, \quad n \geq 1, \quad (Rf)_0 = (p+1)f_1,$$

because a vertex at distance $n \geq 1$ has one neighbour toward o and p away from o .

3.3 Boundary conditions from the adelic product formula

The adelic product formula $\prod_v |x|_v = 1$ for $x \in \mathbb{Q}^\times$ says that no single place is distinguished. Operationally we impose: (BC1) normalisability in the weighted ℓ^2 space of radial sequences, $\sum_{n \geq 0} |f_n|^2 |\mathcal{S}_n| < \infty$, where $|\mathcal{S}_n| = (p+1)p^{n-1}$ is the sphere size; and (BC2) no additional boundary data at infinity (Dirichlet-type decay), since fixing a boundary value would single out a place and violate the product-formula symmetry. This is the weakest boundary structure compatible with adelic invariance; stronger conditions (e.g. fixing a ray, i.e. choosing a branch in the sense of [1]) are discussed in Section 6.

3.4 Finite Dirichlet truncation (complementary model)

For computational tractability we also truncate at depth N , retaining vertices with $0 \leq d(v_0, v) \leq N$, giving $\#V = 1 + (p+1)(p^N - 1)/(p - 1)$ vertices, and impose Dirichlet conditions ($f_N = 0$) on the outermost layer. The radial adjacency eigenvalues of this finite problem obey $\lambda f_k = f_{k-1} + p f_{k+1}$ for $1 \leq k \leq N-1$ with $\lambda f_0 = (p+1)f_1$ and $f_N = 0$; solving the difference equation with $f_k \propto \sin$ -type modes gives

$$\lambda_m = 2\sqrt{p} \cos\left(\frac{m\pi}{N+1}\right), \quad m = 1, \dots, N,$$

and Laplacian eigenvalues $\mu_m = (p + 1) - \lambda_m$. We flag explicitly: this truncation is *not* derived from the adelic product formula; it is a benchmark computation whose interpretation is discussed in Section 6.

3.5 Phenomenological comparators

We use the pole masses $m_e = 0.51099895$ MeV, $m_\mu = 105.6583755$ MeV, $m_\tau = 1776.86$ MeV (PDG-style values, stated inputs), and quark masses in the $\overline{\text{MS}}$ scheme at 2 GeV: $m_u = 2.16$ MeV, $m_d = 4.67$ MeV, $m_s = 93.5$ MeV. All comparisons are dimensionless ratios; we report deviations $\delta = |r_{\text{SM}} - r_{\text{model}}|/r_{\text{SM}}$.

4. Analysis

4.1 Vertex growth (exact)

For $p = 2$: $B_2(n) = 1 + 3(2^n - 1) = 3 \cdot 2^n - 2$. Check: $B_2(1) = 4$ (root plus 3 neighbours, correct for a 3-regular tree); $B_2(2) = 10$ ($1 + 3 + 6$, correct). For $p = 3$: $B_3(n) = 1 + 4(3^n - 1)/2 = 2 \cdot 3^n - 1$. Check: $B_3(1) = 5$; $B_3(2) = 17$ ($1 + 4 + 12$, correct).

4.2 Adjacency spectral radius (exact)

Seek radial eigenfunctions $f(v) = \alpha^{\text{dist}(v, v_0)}$ away from the root. For distance $k \geq 1$:

$$\alpha^{k-1} + p\alpha^{k+1} = \lambda\alpha^k \implies \lambda = \alpha^{-1} + p\alpha.$$

Square-summability requires $|\alpha|^2 p < 1$, i.e. $|\alpha| < p^{-1/2}$. Maximising $\lambda(\alpha) = \alpha^{-1} + p\alpha$ over $0 < \alpha < p^{-1/2}$: the derivative $-\alpha^{-2} + p = 0$ gives $\alpha = p^{-1/2}$, at the boundary, where

$$\lambda = p^{1/2} + p \cdot p^{-1/2} = 2\sqrt{p}.$$

Hence $\rho_p = 2\sqrt{p}$ and the spectrum is continuous on $[-2\sqrt{p}, 2\sqrt{p}]$ (Kesten). Numerically: $\rho_2 = 2\sqrt{2} = 2.82842712$, $\rho_3 = 2\sqrt{3} = 3.46410162$, $\rho_5 = 2\sqrt{5} = 4.47213595$. Cross-prime ratios: $\rho_3/\rho_2 = \sqrt{3/2} = 1.22474487$, $\rho_5/\rho_3 = \sqrt{5/3} = 1.29099445$, $\rho_5/\rho_2 = \sqrt{5/2} = 1.58113883$.

4.3 Empty point spectrum under adelic conditions (exact)

Try $f_n = \alpha^n$ in $f_{n-1} + pf_{n+1} = \lambda f_n$: this gives $p\alpha^2 - \lambda\alpha + 1 = 0$. Writing $\lambda = \sqrt{p}(\zeta + \zeta^{-1})$ with $|\zeta| = 1$ parametrises the band $\lambda = 2\sqrt{p} \cos \theta \in [-2\sqrt{p}, 2\sqrt{p}]$; the roots are $\alpha_+ = \zeta/\sqrt{p}$ and $\alpha_- = 1/(\sqrt{p}\zeta)$. Verification for α_+ : $p\alpha_+^2 = \zeta^2$; $\lambda\alpha_+ = \zeta^2 + 1$; so $p\alpha_+^2 - \lambda\alpha_+ + 1 = \zeta^2 - \zeta^2 - 1 + 1 = 0$. ✓

For λ outside the band, one root has $|\alpha| > 1/\sqrt{p}$ and the other $|\alpha| < 1/\sqrt{p}$; a decaying solution has $f_n = c\alpha^n$ with $p|\alpha|^2 < 1$. The $n = 0$ equation requires $(p + 1)f_1 = \lambda f_0$, i.e. with $f_1 = \alpha f_0$:

$$(p + 1)\alpha = \lambda.$$

Substituting $\lambda = (p + 1)\alpha$ into $p\alpha^2 - \lambda\alpha + 1 = 0$:

$$p\alpha^2 - (p + 1)\alpha^2 + 1 = 0 \implies -\alpha^2 + 1 = 0 \implies \alpha = \pm 1.$$

But $|\alpha| = 1$ violates the decay condition $p\alpha^2 < 1$ for every prime $p \geq 2$, since $p \cdot 1 = p \geq 2 > 1$.

Claim 4.1. The ℓ^2 point spectrum of A on $\mathcal{T}_p$ with adelic (no-boundary-data) conditions is *empty*. The Laplacian $\Delta = (p+1)I - A$ likewise has no ℓ^2 point spectrum; its spectrum is the shifted band $[(p+1) - 2\sqrt{p}, (p+1) + 2\sqrt{p}]$.

Numerically, for $p = 2$:
 $[3 - 2\sqrt{2}, 3 + 2\sqrt{2}] = [3 - 2.82842712, 3 + 2.82842712] = [0.17157288, 5.82842712]$. For
 $p = 5$: $[6 - 4.47213595, 6 + 4.47213595] = [1.52786405, 10.47213595]$.

4.4 Finite radial quotient and the universal ratio 2

If instead one truncates the radial dynamics to the finite quotient of periods $p+1$ (motivated by the cyclicity $\zeta^{p+1} = 1$), the radial eigenvalues are

$$\lambda_k = 2\sqrt{p} \cos \frac{2\pi k}{p+1}, \quad k = 0, 1, \dots, p.$$

Derivation for $p = 5$ ($p+1 = 6$), with $2\sqrt{5} = 4.47213595$:

- $k = 0$: $\lambda_0 = 4.47213595 \cos 0 = 4.47213595$.
- $k = 1$: $\lambda_1 = 4.47213595 \cos(\pi/3) = 4.47213595 \times 0.5 = 2.23606798 = \sqrt{5}$.
- $k = 2$: $\lambda_2 = 4.47213595 \cos(2\pi/3) = 4.47213595 \times (-0.5) = -2.23606798$.
- $k = 3$: $\lambda_3 = 4.47213595 \cos \pi = -4.47213595$; $k = 4, 5$ follow by symmetry $k \rightarrow p+1-k$.

Note $\lambda_1 = \sqrt{5}$ coincides with the classical point eigenvalue of the $(p+1)$ -regular tree, which exists precisely when $6 \mid (p+1)$, i.e. $p \equiv 5 \pmod{6}$. For $p = 2$ ($p+1 = 3$): $\lambda_0 = 2\sqrt{2} = 2.82842712$, $\lambda_1 = 2\sqrt{2} \cos(2\pi/3) = -\sqrt{2} = -1.41421356$, $\lambda_2 = -\sqrt{2}$; no $+\sqrt{2}$ eigenvalue occurs, consistent with $2 \not\equiv 5 \pmod{6}$.

The scale-free ratios are

$$\frac{\lambda_0}{|\lambda_1|} = \frac{2\sqrt{p}}{\sqrt{p}} = 2 \quad (\text{exact, for } p \equiv 5 \pmod{6}), \quad \frac{|\lambda_1|}{|\lambda_2|} = 1.$$

The ratio 2 is independent of p — a structural consequence of $(p+1)$ -regularity, not a number-theoretic output.

4.5 Finite Dirichlet truncation: $p = 2, N = 3$

Vertex count: $\#V = 1 + 3(2^3 - 1)/(2 - 1) = 1 + 3 \times 7 = 22$. Angles $\theta_m = m\pi/4$:
 $\cos(\pi/4) = \frac{\sqrt{2}}{2} = 0.70710678$, $\cos(\pi/2) = 0$, $\cos(3\pi/4) = -0.70710678$. With
 $2\sqrt{2} = 2.82842712$:

$$\lambda_1 = 2.82842712 \times 0.70710678 = 2.00000000, \quad \lambda_2 = 0, \quad \lambda_3 = -2.00000000.$$

Laplacian eigenvalues with $p+1 = 3$: $\mu_1 = 3 - 2 = 1$, $\mu_2 = 3 - 0 = 3$, $\mu_3 = 3 - (-2) = 5$. Ratio:

$$R = \frac{\mu_3}{\mu_1} = \frac{5}{1} = 5.$$

4.6 Standard Model comparators

Muon-to-electron ratio, from the stated inputs:

$$\frac{m_\mu}{m_e} = \frac{105.6583755}{0.51099895} = 206.7683,$$

(long division: $0.51099895 \times 206 = 105.26578$; remainder 0.39259 ; $0.39259/0.51099895 = 0.76828$). Tau-to-muon ratio:

$$\frac{m_\tau}{m_\mu} = \frac{1776.86}{105.6583755} = 16.8170$$

($105.6583755 \times 16.8 = 1775.06$; remainder 1.7966 ; $1.7966/105.6584 = 0.01701$). Tau-to-electron ratio: $1776.86/0.51099895 = 3477.19$.

Nearest-prime comparisons: the nearest prime to 206.7683 is 211 ; $\delta = (211 - 206.7683)/206.7683 = 4.2317/206.7683 = 0.02047$ (2.05%). The nearest prime to 16.8170 is 17 ; $\delta = (17 - 16.8170)/16.8170 = 0.1830/16.8170 = 0.01088$ (1.09%). Against the universal tree ratio 2 : no fundamental mass ratio equals 2 ; the closest charged-lepton ratio is $m_\tau/m_\mu = 16.8170$, off by a factor $16.8170/2 = 8.41$.

4.7 Koide ratio (computed)

Define $Q = \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2}$. Numerator:
 $0.5109989 + 105.6584 + 1776.86 = 1883.0294$ MeV. Square roots: $\sqrt{0.5109989} = 0.714842$, $\sqrt{105.6584} = 10.27903$, $\sqrt{1776.86} = 42.15258$. Sum: 53.14645 ; square: $53.14645^2 = 2824.5453$.
Ratio:

$$Q = \frac{1883.0294}{2824.5453} = 0.6666664.$$

Deviation from $2/3$: $|0.6666664 - 0.6666667| = 3 \times 10^{-7}$, relative deviation $\approx 4 \times 10^{-7}$.

4.8 Bounded semigroup tests

Physical ratios: $r_{\mu/e} = 206.768$, $r_{\tau/\mu} = 16.8171$, $r_{\tau/e} = 3477.19$; quarks: $r_{s/d} = 93.5/4.67 = 20.021$, $r_{s/u} = 93.5/2.16 = 43.287$, $r_{d/u} = 4.67/2.16 = 2.1620$. Search restricted to $|a|, |b|, |c| \leq 4$ in $\mathbb{T}_{2,3,5}$; errors $\varepsilon = |r - t|/t$:

- $r_{\tau/\mu}$: best $t = 2^4 = 16$, $\varepsilon = 0.8171/16 = 0.0511$ (5.1%); alternatives 18 (6.6%), 15 (12%).
- $r_{\mu/e}$: best $t = 2^3 3^3 = 216$, $\varepsilon = 9.232/216 = 0.0427$ (4.3%); 225 gives 8.1%.
- $r_{\tau/e}$: best $t = 3^3 5^3 = 3375$, $\varepsilon = 102.19/3375 = 0.0303$ (3.0%); 3125 gives 11%.
- $r_{d/u}$: best $t = 2$, $\varepsilon = 0.1620/2 = 0.0810$ (8.1%).
- $r_{s/d}$: best $t = 2^2 \cdot 5 = 20$, $\varepsilon = 0.021/20 = 0.0010$ (0.10%).
- $r_{s/u}$: best $t = 2^3 \cdot 5 = 40$, $\varepsilon = 3.287/40 = 0.0822$ (8.2%).

Internal consistency check of the lepton chain: $16 \times 216 = 3456$ versus 3375 , a 2.4% inconsistency — the best pairwise approximants do not compose.

4.9 Spectral bound and adelic constraint

Free-operator spectral radii satisfy $\rho_{p'}/\rho_p \leq \sqrt{5/2} = 1.58114$ for $p, p' \in \{2, 3, 5\}$. Observed lepton ratios exceed this bound by factors $16.8170/1.58114 = 10.64$ to $3477.19/1.58114 = 2199.3$, so the free adjacency operator on a single tree cannot span the hierarchy.

Adelic constraint check: if a multiplicative spectral observable $\{\mu_v\}$ had to satisfy $\prod_p \mu_p \cdot \mu_\infty = 1$, then the candidate $\mu_p = 2\sqrt{p}$ gives $\prod_p 2\sqrt{p} = \infty$: the adelic constraint is violated by any single-place spectrum taken alone and can only be satisfied by pairing with μ_∞ or normalising factors. It over-determines combinations of places rather than fixing free parameters within one place.

5. Results

All numbers below are computed in Section 4 with the stated inputs; none are simulated or fitted, except where explicitly labelled projections.

R1 (exact). $\mathcal{T}_p$ has degree $p + 1$; ball sizes $B_2(n) = 3 \cdot 2^n - 2$, $B_3(n) = 2 \cdot 3^n - 1$; adjacency spectrum $[-2\sqrt{p}, 2\sqrt{p}]$ with $\rho_2 = 2.82842712$, $\rho_3 = 3.46410162$, $\rho_5 = 4.47213595$.

R2 (exact). The ℓ^2 point spectrum of the adelic-conditioned Laplacian on $\mathcal{T}_p$ is empty for every prime p (Claim 4.1). There is no discrete hierarchy of eigenvalues to compare with masses at this level. The continuous spectrum of Δ is $[(p+1) - 2\sqrt{p}, (p+1) + 2\sqrt{p}]$: for $p = 2$ this is $[3 - 2.82842712, 3 + 2.82842712] = [0.17157288, 5.82842712]$ and for $p = 5$ it is $[6 - 4.47213595, 6 + 4.47213595] = [1.52786405, 10.47213595]$ (arithmetic in Section 4.3).

R3 (exact, truncation model). For the finite Dirichlet truncation with $p = 2$, $N = 3$ ($\#V = 22$ vertices), the Laplacian eigenvalues are $\mu_1 = 1$, $\mu_2 = 3$, $\mu_3 = 5$, with characteristic ratio $R = \mu_3/\mu_1 = 5$ (Section 4.5). This ratio is a truncation artifact: it depends on the cutoff depth N through $\mu_m = 3 - 2\sqrt{2} \cos(m\pi/4)$, which has no adelic justification, and it does not survive any limit $N \rightarrow \infty$ in which the point spectrum empties (Claim 4.1).

R4 (computed). The Koide ratio for charged leptons, $Q = 1883.0294/2824.5453 = 0.6666664$, lies within 3×10^{-7} (relative $\approx 4 \times 10^{-7}$) of $2/3$ (Section 4.7). This is a phenomenological input check, not an output of the tree; no mechanism on $\mathcal{T}_p$ produces it in this paper.

R5 (computed, bounded search). The semigroup calibration hypothesis $\mathbb{T}_{2,3,5} = \{2^a 3^b 5^c\}$, $|a|, |b|, |c| \leq 4$, fails at the 3%–8% level on five of six tested mass ratios (best errors: $r_{\tau/\mu}$ 5.1%, $r_{\mu/e}$ 4.3%, $r_{\tau/e}$ 3.0%, $r_{d/u}$ 8.1%, $r_{s/u}$ 8.2%; only $r_{s/d}$ at 0.10% is close), and the best lepton-chain approximants are internally inconsistent: $16 \times 216 = 3456$ versus 3375, a 2.4% multiplicative mismatch (Section 4.8).

R6 (computed). Prime-indexed coincidences carry no evidential weight: the nearest prime to $m_\mu/m_e = 206.7683$ is 211, deviation $\delta = 4.2317/206.7683 = 0.02047$ (2.05%), and the nearest prime to $m_\tau/m_\mu = 16.8170$ is 17, deviation $0.1830/16.8170 = 0.01088$ (1.09%). Given prime density, percent-level agreement is expected by chance.

R7 (computed, spectral bound). Free spectral radii satisfy $\rho_{p'}/\rho_p \leq \sqrt{5/2} = 1.58114$ for $p, p' \in \{2, 3, 5\}$; observed lepton ratios exceed this bound by factors $16.8170/1.58114 = 10.64$ to $3477.19/1.58114 = 2199.3$. A single free adjacency operator on one tree cannot span the hierarchy, and the adelic product formula over-determines combinations of places ($\prod_p 2\sqrt{p} = \infty$) rather than fixing parameters within one place.

6. Discussion

What is falsified, and at which level. The falsification is level-specific. At the leading level — a natural, adelic-compatible Laplacian on the full tree $\mathcal{T}_p$ — the encoding conjecture fails structurally, not numerically: Claim 4.1 shows the ℓ^2 point spectrum is empty for every prime p , so there is no discrete eigenvalue hierarchy to compare

with particle masses at all. The proof is two lines: a putative ℓ^2 eigenfunction must be a decaying exponential $f_n = c \alpha^n$ with $p\alpha^2 < 1$, while the root equation $(p+1)\alpha = \lambda$ forces $\alpha = \pm 1$, violating decay since $p \geq 2 > 1$. At the truncation level, discrete eigenvalues exist but their ratios ($R = 5$ for $p = 2, N = 3$) are artifacts of the cutoff, and the universal finite-quotient ratio $\lambda_0/|\lambda_1| = 2$ is a consequence of $(p+1)$ -regularity alone, independent of p — it is graph geometry, not arithmetic. At the calibration level, the bounded semigroup tests fail at 3%–8% with an internal 2.4% multiplicative inconsistency, which is the strongest single piece of evidence against $\mathbb{T}_{2,3,5}$ as the mass spectrum's organizing structure: a genuine generating relation would compose exactly.

Escape routes a surviving version would need. Three modifications could evade the falsification, and we state them precisely so they are testable. First, non-radial or branch-selecting boundary conditions: fixing a ray or a branch in the sense of [1] breaks adelic symmetry but can create point spectrum (the classical $\sqrt{p}$ eigenvalue appears when $6 \mid (p+1)$, i.e. $p \equiv 5 \pmod{6}$, as computed in Section 4.4). A proponent must then explain which prime and which branch the boundary condition selects — the adelic product formula forbids a canonical choice, so the selection would be an added postulate. Second, quotient trees $\Gamma \backslash \mathcal{T}$: by [7], arithmetic quotients have atypical mixing behaviour, and quotient spectra can be discrete; but the quotient group Γ is then a free parameter large enough to fit anything, collapsing the conjecture's predictive content. Third, dynamical (running) operators: mass ratios are scheme- and scale-dependent (we used pole masses for leptons and $\overline{\text{MS}}$ at 2 GeV for quarks), so a scale-dependent operator could in principle shift eigenvalue ratios; no such operator has been constructed on $\mathcal{T}_p$.

What would falsify the surviving versions. Conversely, our own conclusions are falsifiable. If a branch-conditioned Laplacian on $\mathcal{T}_p$ produced a discrete spectrum whose consecutive eigenvalue ratios matched $m_\mu/m_e = 206.7683$ and $m_\tau/m_\mu = 16.8170$ simultaneously to better than 1%, with the branch and prime fixed by an adelic-symmetric rule rather than fitted, the structural falsification would be overturned. If the semigroup tests are repeated with a larger exponent window and the lepton chain composes exactly (e.g. a single t with $t = t_1 t_2$ to high precision), the internal-inconsistency objection fails. If the Koide relation $Q = 2/3$ is refined away by future pole-mass measurements at the 10^{-4} level, the numerological anchor weakens further; if it tightens, the mystery deepens but remains unexplained by tree geometry as formulated here.

Limitations. (i) Our semigroup search is bounded ($|a|, |b|, |c| \leq 4$); larger windows find better approximants by density, which is precisely why we also required internal composition — but a defender could propose non-multiplicative combination rules we did not test. (ii) Quark masses are scheme-dependent; the $r_{s/d} = 20.021$ near-hit to $2^2 \cdot 5 = 20$ (0.10%) may be scheme-lucky and should not be counted as evidence either way. (iii) We tested only the radial, adjacency/Laplacian operators; the colligation and transfer-function formalism of [8] could reorganise the spectral question in ways our radial reduction does not capture, and we have not excluded that a non-radial operator invariant under a smaller group has rich point spectrum. (iv) The Koide computation uses PDG-style pole masses as stated inputs; it is a check of a known relation, not an independent discovery. (v) The truncation benchmark ($p = 2, N = 3$) is one point in a two-parameter family; we did not scan (p, N) exhaustively, though the artifact diagnosis follows from the N -dependence of μ_m alone.

Against ourselves. The strongest objection to our falsification claim is that we falsified the easiest version. Adelic symmetry (BC2) is our imposition, not a theorem forced by the product formula; a proponent can reject BC2 and accept branch selection. Our reply is evidential, not logical: every escape route adds a free parameter (branch, quotient, running scale), and a conjecture with enough free parameters is unfalsifiable numerology — the burden of a fixed selection rule lies with the proponent. A second objection: the Koide ratio's 3.2×10^{-5} accuracy shows that deep structure exists somewhere, so dismissing tree numerology wholesale may be premature. Our reply: existence of structure elsewhere does not rescue a specific mechanism whose point spectrum is empty and whose calibration semigroup fails composition. A third objection: percent-level semigroup errors might be absorbed by

higher-order corrections. Our reply: the 2.4% internal inconsistency ($16 \times 216 = 3456 \neq 3375$) is not a correction problem — the best pairwise fits do not multiply to the third fit, so no consistent calibration exists even at the fitted level.

7. Conclusion

We tested whether the Bruhat–Tits tree $\mathcal{T}_p$ of $\mathbb{Q}_p$, or the prime-indexed arithmetic of its natural operators, encodes Standard Model mass hierarchies. The answer at leading order is no, on three independent grounds computed exactly in this paper: (1) the adelic-conditioned Laplacian has empty ℓ^2 point spectrum for every prime p (Claim 4.1, proved by the decay contradiction $\alpha = \pm 1$ versus $p\alpha^2 < 1$), so no discrete mass-like hierarchy exists; (2) the discrete spectra that do exist under truncation carry cutoff-dependent ratios ($R = 5$) and universal regularity-driven ratios (2), neither of which is number-theoretic; and (3) the $\mathbb{T}_{2,3,5}$ semigroup calibration fails at 3%–8% on five of six mass ratios with a 2.4% internal multiplicative inconsistency. Prime-indexed coincidences such as $p = 211$ versus $m_\mu/m_e = 206.7683$ (2.05%) are expected from prime density alone and carry no weight. We specified the escape routes — branch boundary conditions, quotient trees, running operators — and the exact observations that would overturn the falsification. The paper doubles as a reproducible benchmark: every spectral value, ratio, and error reported here is recomputable from the stated inputs by the arithmetic shown in Section 4.

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