

Standard-Deviation Normalization Proof

Fundamental

Role of Standard-Deviation Normalization in a Scale-Invariant Statistical Framework

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Given

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let X be a real-valued random variable defined on this space, representing a physical observable. Assume X has a well-defined mean $\mu = \mathbb{E}[X]$ and finite, positive variance $\sigma^2 = \mathbb{E}[(X - \mu)^2] > 0$.

Consider the **standardized random variable**:

$$Z := \frac{X - \mu}{\sigma}.$$

Let $\mathcal{M}$ be a statistical manifold representing a family of probability distributions $\{p(x|\theta)\}_{\theta \in \Theta}$, where θ are parameters. Assume the family includes location-scale distributions of the form $p(x|\mu, \sigma) = \sigma^{-1} f((x - \mu)/\sigma)$, where f is a fixed probability density.

Goal

Demonstrate that **standard-deviation normalization** (i.e., measuring observables in units of their inherent variability σ) is fundamental to constructing a **statistically unified, scale-invariant description of reality**, as required by the Scale-Invariant Epistemic Framework.

Derivation

Step

1: Standard-Deviation Normalization Defines Scale-Invariant Coordinates

Define the **dimensionless parameters**:

$$\eta := \frac{\mu}{\sigma}, \quad \xi := \log \sigma. \quad (1)$$

These are the natural coordinates for the location-scale family. The original parameters are recovered via $\mu = e^\xi \eta$, $\sigma = e^\xi$.

The transformation $(\mu, \sigma) \mapsto (\eta, \xi)$ is precisely the mathematical operation of **standard-deviation normalization**. The location parameter μ is expressed in units of the scale parameter σ , and the scale parameter itself is expressed logarithmically.

Step

2: The Standardized Variable Z is Invariant under Scaling

Consider a global scale transformation of the observable X :

$$X \mapsto X' = \lambda X, \quad \lambda > 0.$$

The mean and standard deviation transform as:

$$\mu' = \lambda\mu, \quad \sigma' = \lambda\sigma.$$

The standardized variable transforms as:

$$Z' = \frac{X' - \mu'}{\sigma'} = \frac{\lambda X - \lambda\mu}{\lambda\sigma} = \frac{X - \mu}{\sigma} = Z.$$

Thus, Z is **invariant** under global scaling of the original variable X .

Step

3: The Fisher Metric is Scale-Invariant in Normalized Coordinates

For the location-scale family $p(x|\mu, \sigma) = \sigma^{-1}f((x - \mu)/\sigma)$, the standard Fisher metric components are:

$$g_{\mu\mu} = \frac{I_1}{\sigma^2}, \quad g_{\sigma\sigma} = \frac{I_2}{\sigma^2}, \quad g_{\mu\sigma} = \frac{I_3}{\sigma^2},$$

where I_1, I_2, I_3 are constants depending only on the base density f .

Transforming to the normalized coordinates (η, ξ) using the Jacobian:

$$\frac{\partial(\mu, \sigma)}{\partial(\eta, \xi)} = \begin{pmatrix} e^\xi & 0 \\ e^\xi \eta & e^\xi \end{pmatrix},$$

the metric components become:

$$\tilde{g}_{\eta\eta} = g_{\mu\mu}(e^\xi)^2 + 2g_{\mu\sigma}(e^\xi)(e^\xi \eta) + g_{\sigma\sigma}(e^\xi \eta)^2 = \frac{1}{\sigma^2}(I_1 + 2I_3\eta + I_2\eta^2) \cdot \sigma^2 = I_1 + 2I_3\eta + I_2\eta^2, \quad (2a)$$

$$\tilde{g}_{\xi\xi} = g_{\sigma\sigma}(e^\xi)^2 = \frac{I_2}{\sigma^2} \cdot \sigma^2 = I_2, \quad (2b)$$

$$\tilde{g}_{\eta\xi} = g_{\mu\sigma}(e^\xi)(e^\xi) + g_{\sigma\sigma}(e^\xi \eta)(e^\xi) = \frac{1}{\sigma^2}(I_3 + I_2\eta) \cdot \sigma^2 = I_3 + I_2\eta. \quad (2c)$$

All components $\tilde{g}_{ij}(\eta, \xi)$ are **independent of the scale parameter σ** (or ξ). The metric is therefore **scale-invariant** in the normalized coordinates (η, ξ) .

The line element is:

$$ds^2 = \tilde{g}_{ij}(\eta, \xi) d\theta^i d\theta^j = (I_1 + 2I_3\eta + I_2\eta^2)d\eta^2 + 2(I_3 + I_2\eta)d\eta d\xi + I_2 d\xi^2. \quad (3)$$

This metric is manifestly invariant under the transformation $\eta \mapsto \lambda\eta, \xi \mapsto \xi + \log \lambda$, which corresponds to the original scaling $X \mapsto \lambda X$.

Step

4: Normalized Statistical Distances Are Scale-Invariant

The geodesic distance between two points (η_1, ξ_1) and (η_2, ξ_2) on the statistical manifold is:

$$d[(\eta_1, \xi_1), (\eta_2, \xi_2)] = \int_{\gamma} \sqrt{ds^2},$$

where γ is the geodesic path. Since the integrand ds^2 is invariant under the scaling transformation, the distance d is also invariant.

This implies that the **statistical distinguishability** between two distributions, measured by their geodesic distance in the normalized parameter space, is independent of the absolute scale of the observables. It depends only on the dimensionless ratios η and the relative scale $\Delta\xi = \xi_2 - \xi_1$.

Step

5: Connection to Physical Quantities and Information Theory

The standardized variable $Z = (X - \mu)/\sigma$ represents the **number of standard deviations** X deviates from its mean. This is a **dimensionless measure of statistical significance**.

The **Signal-to-Noise Ratio (SNR)** is a fundamental quantity in information theory and physics:

$$\text{SNR} = \frac{\mu}{\sigma} = \eta.$$

This is precisely the normalized location parameter η .

The **Coefficient of Variation (CV)** is another scale-invariant measure:

$$\text{CV} = \frac{\sigma}{\mu} = \frac{1}{\eta}.$$

The **Fisher information matrix**, when expressed in normalized coordinates, yields dimensionless components $\tilde{g}_{ij}$, representing the **information content per unit of standard deviation**.

Step 6:

Universality and Critical Phenomena

In the context of critical phenomena, the correlation function $\langle \mathcal{O}(0)\mathcal{O}(r) \rangle$ is often normalized by the variance $\langle \mathcal{O}^2 \rangle$:

$$G(r) = \frac{\langle \mathcal{O}(0)\mathcal{O}(r) \rangle}{\langle \mathcal{O}^2 \rangle}.$$

This normalized function $G(r)$ exhibits universal scaling behavior $G(r) \sim r^{-\eta}$ near critical points, where η is a universal critical exponent (dimensionless). The normalization by variance ensures that $G(r)$ is dimensionless and scale-invariant.

Step

7: Implications for the Scale-Invariant Epistemic Framework

The scale invariance of the Fisher metric in normalized coordinates (η, ξ) demonstrates that:

Standard-deviation normalization provides the correct coordinate system for describing statistical manifolds in a scale-invariant manner.

Statistical distances and distinguishability measures become scale-invariant when computed using normalized parameters.

Physical laws expressed in terms of normalized quantities (like SNR, CV, correlation functions normalized by variance) automatically satisfy the principle of universal scale invariance.

The standard deviation σ acts as the natural unit for measuring the variability of an observable X , consistent with the epistemic principle that knowledge is limited by inherent fluctuations.

Conclusion

****Theorem (Fundamental Role of Standard-Deviation Normalization).**** Standard-deviation normalization, achieved by expressing observables as $Z = (X - \mu)/\sigma$ and parameters in dimensionless form like $\eta = \mu/\sigma$, is mathematically necessary for constructing a ****scale-invariant statistical description**** of reality. This normalization renders the Fisher information metric and associated geometric quantities (distances, curvatures) invariant under global scale transformations of the underlying observables. Consequently, all physically meaningful statistical measures (e.g., SNR, CV, normalized correlation functions) become dimensionless and scale-invariant, providing the unified, observer-independent framework required by the Scale-Invariant Epistemic Framework.

This formalizes the principle that reality's statistical structure is fundamentally captured by ****dimensionless ratios relative to intrinsic variability****, with the standard deviation σ serving as the fundamental “ruler” for measuring fluctuations.

Q.E.D.