

The Algebraic Origin of Zitterbewegung: A Hierarchy of Ownership from Clifford Structure to Observed Oscillation

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Abstract

Zitterbewegung (ZB) — the rapid trembling of a relativistic quantum particle's position — is usually presented as a kinematic curiosity of the Dirac equation, and simulator experiments in cold atoms and engineered Dirac-like systems must interpret observed oscillations without a criterion for when ZB must appear. The specific gap addressed here is the absence of a representation-independent statement of why the oscillation exists: whether it is a consequence of the particular 4×4 Dirac matrices, of the geometry of spacetime, or of an algebraic fact that survives in any dimension and any representation. We formulate a hierarchy of "ownership" — algebra, calculus, trigonometry, geometry — and show, by explicit operator derivation on a Clifford module, that the oscillation is owned by the algebra: the anticommutator $\{H, \alpha_j\} = 2cp_j$ forces the velocity operator to evolve as $\alpha_j(t) = cp_j H^{-1} + Z_j e^{-2iHt/\hbar}$, producing interference between positive- and negative-energy eigenspaces at the beat frequency $\omega_{\text{ZB}} = (E_+ - E_-)/\hbar$. We generalize the construction to arbitrary Clifford algebras and to any Hamiltonian whose velocity operators fail to commute with it through Clifford structure. Numerically, the rest-frame electron oscillation frequency is $f_0 = 2.471 \times 10^{20}$ Hz with position amplitude 1.931×10^{-13} m, and an illustrative gapped Dirac material with band edges at $\pm\Delta$, $\Delta = 0.1$ eV, gives $f = 4.836 \times 10^{13}$ Hz. The result converts ZB from a Dirac-equation artifact into a spectral diagnostic: suppressing one eigenspace eliminates the oscillation, which is directly testable in existing simulator platforms.

1. Introduction

Teams running quantum simulators of relativistic dynamics — cold-atom laboratories implementing Dirac-like laser couplings, and groups engineering spin-orbit-coupled materials or trapped-ion analogues — routinely observe rapid oscillatory motion of their wave packets and must decide whether that motion is the genuine Zitterbewegung phenomenon, an artifact of their drive scheme, or a controllable resource. This decision matters in practice. If the oscillation is on resonance with an applied modulation, it can be converted into directed center-of-mass motion, stopped, or re-launched, as demonstrated for Dirac-like equations in the analysis of arXiv:1105.2884 [1]; cold-atom implementations show that external fields can suppress or enhance the amplitude and shift the frequencies [2], and that driven mirror oscillation can control amplitude, frequency, and damping [4]. Today, however, the interpretation of these oscillations rests on familiarity with one specific equation — the 4×4 free Dirac Hamiltonian in the standard representation. A practitioner working with a spin-1 cold-atom system [2], a tripod-scheme laser system [4], or a two-band condensed-matter analogue has no general criterion telling them whether an observed oscillation is algebraically forced or incidental to their model.

The cost of this gap is concrete: experimental design proceeds by analogy rather than by principle. A general criterion would change simulator practice in two ways. First, it would tell an experimentalist in advance which degrees of freedom must be populated for the oscillation to exist — namely, coherent superpositions across two eigenspaces of the Hamiltonian — so that state preparation can either guarantee or deliberately suppress ZB. Second, it would identify the oscillation frequency as a spectroscopic observable: a direct measurement of the inter-eigenspace energy splitting, independent of the representation used to engineer the Hamiltonian.

This paper addresses that gap with a hierarchy-of-ownership conjecture, stated up front so that its status is clear. We conjecture that Zitterbewegung oscillation is fundamentally an algebraic/spectral fact: it arises whenever a velocity operator fails to commute with the Hamiltonian because of Clifford-algebraic structure, independent of dimension or representation. Calculus merely produces the solution of the resulting operator differential equation; trigonometry is only the notation in

which the solution is written; geometry is a visualizable shadow of an interference phenomenon that exists without any picture. The claim is testable: if the algebraic account is right, then removing the negative-energy (or, generally, the second-eigenspace) component of a wave packet must eliminate the oscillation, in any platform, at any dimension.

The contributions are:

- A closed-form operator derivation of the velocity and position dynamics, $\alpha_j(t) = cp_j H^{-1} + Z_j e^{-2iHt/\hbar}$ and the corresponding position decomposition, from the single algebraic input $\{H, \alpha_j\} = 2cp_j$, with the anticommutator shown to be a Clifford identity (Section 3).
- A generalization theorem: for any Hamiltonian on a Clifford module whose velocity operators are Clifford generators, the oscillation exists exactly when two eigenspaces are coherently populated, and its frequency always equals the inter-eigenspace beat frequency $(E_+ - E_-)/\hbar$ (Section 3, proved in Section 4).
- Fully worked numerical evaluations of the rest-frame electron case ($f_0 = 2.471 \times 10^{20}$ Hz, amplitude 1.931×10^{-13} m) and of an illustrative gapped Dirac material ($f = 4.836 \times 10^{13}$ Hz at $\Delta = 0.1$ eV with band edges at $\pm\Delta$), with every arithmetic step shown (Section 4).
- A falsifiable experimental signature — eigenspace-filtered wave-packet dynamics — and an assessment of readiness, blockers, and the smallest next test on existing simulator platforms (Sections 6 and 7).

2. Background and Related Work

The Zitterbewegung phenomenon has been studied in several different contexts as the jittering center-of-mass motion predicted by free-space Dirac or Dirac-like equations. The work of arXiv:1105.2884 [1] shows that ZB can be converted into directed center-of-mass motion by a modulation of the Dirac-like equation when the modulation is on resonance with the ZB frequency, and that tailored modulation may also stop or re-launch the motion. This is direct evidence that the oscillation frequency is an operationally accessible and exploitable parameter of a simulator, which is precisely the quantity our algebraic analysis identifies as a beat frequency. The summary supplied for [1] is truncated and gives no further technical detail beyond these statements.

On the experimental side, arXiv:1210.5030 [2] investigates the Zitterbewegung effect in spin-orbit-coupled spin-1 ultracold atoms in the presence of a Zeeman field and a harmonic trap, showing that these external elements significantly affect the oscillatory behavior: the Zeeman field can suppress or enhance the ZB amplitude and change the oscillation frequencies, with a slower dynamics noted in the truncated summary. That ZB survives in a spin-1 system — a different spin dimensionality from the spin- $\frac{1}{2}$ Dirac case — is consistent with the central claim of this paper that the oscillation is not owned by the specific 4×4 Dirac matrices. arXiv:1003.3074 [4] considers mirror oscillation in a "tripod-scheme" laser-atom system and shows, analytically and numerically, that mirror oscillation drives an effective spin-orbit interaction and thereby controls the amplitude, frequency, and damping of cold-atom ZB. Together, [2] and [4] establish that ZB is a controllable, platform-portable phenomenon; what neither supplies, within the summaries available, is a representation-independent account of why it occurs — the gap this paper fills.

A broad research program attributes structural features of physics to algebra rather than to dynamics or geometry, and our hierarchy conjecture places ZB inside that program. arXiv:2009.08489 [3] argues that quantum probabilities and classical probabilities have different origins: classical probabilities always result from an assumed probability measure, while quantum probabilities include transition probabilities with a purely algebraic origin, with a general definition of transition probabilities whose supplied summary truncates before details. This supports the methodological stance that quantum phenomena can be owned by algebra at a level deeper than the calculus used to solve for them. In high-energy theory, arXiv:hep-th/0307046 [5] describes an algebraic construction, via a procedure called twining that breaks the quasi-bialgebra structure of universal enveloping algebras of semi-simple Lie algebras, aimed at categories supporting a state-confinement principle; the summary supplied gives no further detail, and we cite it only as an instance of algebra-first explanation of a physical phenomenon. arXiv:2105.02115 [6] shows that for central potentials, both the fourth component of a Lorentz vector and a Lorentz scalar in the Dirac Hamiltonian give rise, through the conserved Dirac spin-orbital matrix, to Witten's $N = 2$ superalgebra, with the generators' commutativity with the Dirac Hamiltonian studied under an invariance

requirement whose consequences the truncated summary does not state. This is the closest structural relative of our result in the list: it finds hidden algebra governing the Dirac Hamiltonian, though for potentials rather than for the kinetic velocity operators.

Further afield, the algebraic-origin program spans integrable systems and field theory. arXiv:hep-th/9408104 [7] presents an affine $sl(n+1)$ algebraic construction of the basic constrained KP hierarchy, analyzed via two equivalent approaches — a linear matrix eigenvalue problem on a hermitian symmetric space and a constrained KP Lax formulation — and identifies the model as the generalized non-linear Schrödinger hierarchy. arXiv:hep-th/0007032 [8] takes a combinatorial formula of G.-C. Rota and J. A. Stein to perform Wick re-ordering in quantum field theory, so that Wick's theorem becomes a Hopf-algebraic identity called Cliffordization; the appearance of a Clifford-type structure as an efficient computational device in field theory is a useful precedent for our claim that Clifford structure, not the specific Dirac representation, carries the physical content. arXiv:0809.1086 [9] treats the Lie-algebraic origin of metric 3-algebras in three-dimensional superconformal Chern-Simons theories, noting that many such theories admit reformulation as standard gauge theory coupled to matter in which the 3-algebra does not appear explicitly — a cautionary parallel, since it shows that an algebraic formulation can be a choice of description rather than a uniqueness claim; our hierarchy conjecture must therefore be argued for, not assumed. arXiv:1712.04422 [10] formulates, in terms of separation of variables, a generalization of the observation (from O. Babelon and M. Talon, 2002, as cited in its summary) that a plane algebraic curve whose Newton polygon contains d lattice points can be given by d points it passes through, with the coefficients Poisson-commuting when regarded as functions of those points.

The program has known limits, which we take seriously. arXiv:1306.0174 [11] shows that there exist simple groups with a spherical BN-pair of rank 2 which are non-Moufang and hence not of algebraic origin. This is a reminder that "algebraic origin" is a substantive property that can fail; our falsification criterion in Section 6 is designed so that a platform exhibiting ZB without the two-eigenspace interference structure would refute the conjecture in the same way.

Finally, three QNFO corpus entries sit adjacent to this work. The entry "Zitterbewegung: From Archimedean Puzzle to Adelic Observable" [12] states that the structural underdetermination of the position operator in relativistic quantum mechanics arises from a representation-theoretic origin called the Silent Parameter Principle, that $SU(2)$ and $\mathbb{Z}_2^x$ share identical fusion rules at matching levels, and that a 2-adic ZB frequency deviates from the Archimedean value; the summary truncates mid-sentence and supplies no further detail, so we use it only as evidence that the position operator's ambiguity and the ZB frequency are both recognized as structural, not dynamical, puzzles. The entry "Five Pillars, One Structure" [13] reports that five independent QNFO research programs converge on ultrametric (non-Archimedean) mathematics as the correct state-space geometry for fundamental physics, quantum computation, and optimization; and "The Ultrametric Foundation" [14] argues that positional notation is inherently an ultrametric tree of nested cycles and that the Archimedean line is a derived abstraction. These entries propose a non-Archimedean setting in which the present paper's Archimedean beat-frequency statement would require re-derivation; we flag this as an open question in Section 6 rather than importing their conclusions, since their supplied summaries do not contain the technical machinery needed to do so.

3. Methods

3.1 Setting: the Dirac Hamiltonian as a Clifford-module operator

We work in the Heisenberg picture with $\hbar$ explicit. The free Dirac Hamiltonian is

$$H = c \boldsymbol{\alpha} \cdot \mathbf{p} + \beta mc^2,$$

where $\mathbf{p} = (p_1, p_2, p_3)$ is the momentum operator, m the rest mass, c the speed of light, and α_j, β are Hermitian matrices satisfying the Clifford relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} 1, \quad \alpha_j \beta + \beta \alpha_j = 0, \quad \beta^2 = 1.$$

The single algebraic input of the whole derivation is the anticommutator of the Hamiltonian with a velocity generator. Direct expansion using the Clifford relations gives

$$\{H, \alpha_j\} = H\alpha_j + \alpha_j H = c(\boldsymbol{\alpha} \cdot \mathbf{p} \alpha_j + \alpha_j \boldsymbol{\alpha} \cdot \mathbf{p}) + mc^2(\beta \alpha_j + \alpha_j \beta).$$

The β terms vanish by anticommutation. For the kinetic terms, $\alpha_k \alpha_j + \alpha_j \alpha_k = 2\delta_{jk}$ collapses the sum over k to the single term $2p_j$, so

$$\{H, \alpha_j\} = 2cp_j. \quad (3.1)$$

This is the algebraic fact. It holds for any representation of the Clifford relations — any dimension, any matrix size, any similarity transform — because it was derived from the relations themselves and not from a particular set of matrices.

3.2 The velocity operator and its exact time evolution

The velocity operator is $\dot{x}_j = \frac{i}{\hbar}[H, x_j] = c\alpha_j$; the position operator x_j commutes with $\mathbf{p}$ and β , so this identification is exact and representation-independent. The Heisenberg equation for α_j is

$$\frac{d\alpha_j}{dt} = \frac{i}{\hbar}[H, \alpha_j] = \frac{i}{\hbar}(H\alpha_j - \alpha_j H).$$

Using $\alpha_j H = 2cp_j - H\alpha_j$ from (3.1),

$$\frac{d\alpha_j}{dt} = \frac{i}{\hbar}(2H\alpha_j - 2cp_j) = \frac{2i}{\hbar}(H\alpha_j - cp_j). \quad (3.2)$$

Define the shifted operator

$$Z_j \equiv \alpha_j - cp_j H^{-1}.$$

Since H is a function of α_k and β , and Z_j is a linear combination of Clifford generators minus a term built from H itself, one verifies $Z_j H = \alpha_j H - cp_j = (2cp_j - H\alpha_j) - cp_j = -(H\alpha_j - cp_j) = -HZ_j$: the shifted operator anticommutes with the Hamiltonian. Now propose the ansatz

$$\alpha_j(t) = cp_j H^{-1} + Z_j e^{-2iHt/\hbar}.$$

Differentiate: $\frac{d}{dt}\alpha_j(t) = Z_j \cdot \left(-\frac{2iH}{\hbar}\right) e^{-2iHt/\hbar}$. Because Z_j anticommutes with H , $Z_j H = -H Z_j$, so

$$\frac{d\alpha_j}{dt} = -\frac{2i}{\hbar} Z_j H e^{-2iHt/\hbar} = \frac{2i}{\hbar} H Z_j e^{-2iHt/\hbar} = \frac{2i}{\hbar} H (\alpha_j(t) - cp_j H^{-1}),$$

which is exactly (3.2) with the initial condition $\alpha_j(0) = \alpha_j$. Hence the exact solution is

$$\alpha_j(t) = cp_j H^{-1} + Z_j e^{-2iHt/\hbar} \quad (3.3)$$

with $Z_j = \alpha_j - cp_j H^{-1}$, valid for every representation of the Clifford relations. The first term is a constant "quasi-velocity" fixed by the momentum; the second is a unitary rotation of the initial Clifford generator at the frequency set by the spectrum of H .

3.3 Position dynamics and the interference decomposition

Integrating $\dot{x}_j = c\alpha_j(t)$ from 0 to t :

$$x_j(t) = x_j(0) + c^2 p_j H^{-2} t + \int_0^t c Z_j e^{-2iHt'/\hbar} dt'.$$

The integral evaluates term by term since H^{-1} is constant in time:

$$\int_0^t e^{-2iHt'/\hbar} dt' = \frac{\hbar}{-2iH} (e^{-2iHt/\hbar} - 1) = \frac{i\hbar}{2} H^{-1} (e^{-2iHt/\hbar} - 1),$$

so

$$x_j(t) = x_j(0) + c^2 p_j H^{-2} t + \frac{i\hbar c}{2} H^{-1} Z_j \left(e^{-2iHt/\hbar} - 1 \right). \quad (3.4)$$

The position separates into three parts: a classical uniform drift $c^2 p_j H^{-2} t$, a static shift, and an oscillatory interference term. Only the third part is Zitterbewegung.

3.4 Generalization: the beat-frequency theorem

The key generalization drops the specific Dirac form. Let $\mathcal{M}$ be any module for any Clifford algebra $\text{Cl}(p, q)$, let H be a self-adjoint operator on $\mathcal{M}$ with purely discrete non-degenerate spectrum, and suppose the velocity operators $\dot{x}_j$ are Clifford generators γ_a (or linear combinations of them) satisfying

$$\{H, \gamma_a\} = 2 \text{ (operator independent of } \gamma_a \text{ and } H). \quad (3.5)$$

Condition (3.5) is exactly the structure used in Section 3.2; the derivation of (3.3) used nothing about dimension, matrix size, or the Lorentz signature, only (3.1) and the invertibility of H . Therefore (3.3) holds verbatim with α_j replaced by γ_a .

Now decompose the Hilbert space into the eigenspaces of H . Let P_+ and P_- be the spectral projectors onto the positive- and negative-energy sectors with eigenvalues $E_+ > 0$ and $E_- < 0$ (for the free Dirac case, $E_{\pm} = \pm E(\mathbf{p})$ with $E(\mathbf{p}) = \sqrt{c^2 |\mathbf{p}|^2 + m^2 c^4}$). Then

$$e^{-2iHt/\hbar} = e^{-2iE_+t/\hbar} P_+ + e^{-2iE_-t/\hbar} P_- + \dots,$$

where the ellipsis denotes any further eigenspaces. For a state $|\psi\rangle = |\psi_+\rangle + |\psi_-\rangle$ with components in both sectors, the expectation of the oscillatory term in (3.4) contains the cross term

$$\langle \psi_+ | \frac{i\hbar c}{2} H^{-1} Z_j e^{-2iHt/\hbar} | \psi_- \rangle + \text{h.c.} = A_j e^{-i(E_+ - E_-)t/\hbar} + A_j^* e^{+i(E_+ - E_-)t/\hbar},$$

because the projector structure forces the phase difference $e^{-2iE_+t/\hbar} e^{+2iE_-t/\hbar} = e^{-2i(E_+ - E_-)t/\hbar}$ on the bra side and its conjugate; collecting terms, the observable oscillation is at angular frequency

$$\omega_{\text{ZB}} = \frac{E_+ - E_-}{\hbar} \quad (3.6)$$

— an inter-eigenspace beat frequency. Two consequences follow immediately. First, if the state lies entirely in one eigenspace ($P_+|\psi\rangle = |\psi\rangle$ or $P_-|\psi\rangle = |\psi\rangle$), the cross term vanishes identically and there is no Zitterbewegung: the oscillation is a two-eigenspace interference effect, not a property of a single energy eigenstate. Second, the frequency is a spectroscopic observable: measuring ω_{ZB} measures the gap $E_+ - E_-$ directly, whatever representation or platform realizes H .

For the free Dirac case, $E_+ - E_- = 2E(\mathbf{p})$, so

$$\omega_{\text{ZB}}(\mathbf{p}) = \frac{2E(\mathbf{p})}{\hbar} = \frac{2\sqrt{c^2 |\mathbf{p}|^2 + m^2 c^4}}{\hbar} \geq \frac{2mc^2}{\hbar}, \quad (3.7)$$

with equality at $\mathbf{p} = 0$. The frequently quoted value $2mc^2/\hbar$ is thus the rest-frame minimum of a momentum-dependent beat frequency — a point the hierarchy account makes transparent but the single-equation view obscures.

3.5 The hierarchy of ownership

The derivation assigns each mathematical layer a precise role:

- **Algebra (owner).** The Clifford relations (3.1) force the velocity operator out of the commutant of H ; the two-eigenspace structure of a gapped self-adjoint operator then forces interference. Nothing else is used.
- **Calculus (producer).** The linear operator differential equation (3.2) is solved by the exponential ansatz; calculus generates the solution but contributes no physics.
- **Trigonometry (notation).** The complex exponential $e^{-2iHt/\hbar}$ is a rotation; writing it as $\cos$ and $\sin$ changes nothing.

- **Geometry (shadow).** The "trembling" picture is the visualization of the interference term in (3.4); it is derivable, not fundamental, and it is exactly the part that disappears when one eigenspace is removed.

This hierarchy is a conjecture about generality — that every ZB-type oscillation in every platform is owned at the algebra layer — and Section 4 tests its numerical consequences while Section 6 states what would falsify it.

4. Analysis

Every input number below is stated with its source; every arithmetic step is shown. Physical constants are taken as the standard values: $c = 2.998 \times 10^8$ m/s, $m_e = 9.109 \times 10^{-31}$ kg (electron rest mass), $\hbar = 1.0546 \times 10^{-34}$ J·s, and $\hbar = 6.582 \times 10^{-16}$ eV·s (the same constant in different units; consistency is checked in Step 1). The electron mass and speed of light are standard physical constants used as inputs; they are not measured in this work, and every numerical result below inherits their stated precision (four significant figures).

Step 1 — Consistency of the two values of $\hbar$. The ratio of the two stated values must equal the elementary charge e expressed in joules per electronvolt:

$$\frac{\hbar_{\text{J s}}}{\hbar_{\text{eV s}}} = \frac{1.0546 \times 10^{-34}}{6.582 \times 10^{-16}} = \frac{1.0546}{6.582} \times 10^{-18} = 0.16023 \times 10^{-18} = 1.6023 \times 10^{-19} \text{ J/eV}.$$

This ratio 1.6023×10^{-19} J/eV equals the elementary charge expressed in joules per electronvolt, 1.602×10^{-19} J/eV, to the four significant figures carried. The two unit systems are consistent, so results computed in joules and in electronvolts can be compared directly.

Step 2 — Electron rest energy mc^2 . First compute c^2 :

$$c^2 = (2.998 \times 10^8 \text{ m/s})^2 = 8.988 \times 10^{16} \text{ m}^2/\text{s}^2.$$

Then

$$m_e c^2 = (9.109 \times 10^{-31} \text{ kg}) \times (8.988 \times 10^{16} \text{ m}^2/\text{s}^2) = 81.87 \times 10^{-15} \text{ kg m}^2/\text{s}^2 = 8.187 \times 10^{-14} \text{ J}.$$

In electronvolts, using the conversion from Step 1:

$$m_e c^2 = \frac{8.187 \times 10^{-14} \text{ J}}{1.602 \times 10^{-19} \text{ J/eV}} = 5.111 \times 10^5 \text{ eV} = 0.5111 \text{ MeV}.$$

This matches the standard electron rest energy to the four significant figures carried, confirming the constant set is internally consistent.

Step 3 — Rest-frame ZB frequency f_0 . From (3.7) at $\mathbf{p} = 0$, the angular frequency is $\omega_0 = 2m_e c^2 / \hbar$ and the ordinary frequency is $f_0 = \omega_0 / 2\pi$. First the angular frequency in SI units:

$$\omega_0 = \frac{2 \times 8.187 \times 10^{-14} \text{ J}}{1.0546 \times 10^{-34} \text{ J s}} = \frac{1.6374 \times 10^{-13}}{1.0546 \times 10^{-34}} \text{ s}^{-1} = 1.5526 \times 10^{21} \text{ rad/s}.$$

Then

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1.5526 \times 10^{21}}{6.2832} \text{ Hz} = 2.471 \times 10^{20} \text{ Hz}.$$

Cross-check in electronvolt units:
 $\omega_0 = 2 \times 0.5111 \text{ MeV} / (6.582 \times 10^{-16} \text{ eV s}) = 1.0222 \times 10^6 / (6.582 \times 10^{-16}) = 1.553 \times 10^{21} \text{ rad/s}$
 , agreeing to the four figures carried. The rest-frame oscillation period is $T_0 = 1/f_0 = 4.047 \times 10^{-21} \text{ s}$ — roughly four zeptoseconds, far beyond direct time resolution, which is why the frequency is accessible only through its dynamical consequences or in analogue systems.

Step 4 — Rest-frame position amplitude. The oscillatory term in (3.4) carries the prefactor $\frac{i\hbar c}{2}H^{-1}Z_j$; at $\mathbf{p} = 0$, $H^{-1} = (m_e c^2)^{-1}$ and Z_j is bounded by $\|Z_j\| \leq \|\alpha_j\| + \|cp_j H^{-1}\| = 1$ (the momentum term vanishes at rest), so the natural amplitude scale is the reduced Compton wavelength divided by 2:

$$a_0 = \frac{\hbar}{2m_e c} = \frac{1.0546 \times 10^{-34} \text{ J s}}{2 \times (9.109 \times 10^{-31} \text{ kg}) \times (2.998 \times 10^8 \text{ m/s})}.$$

The denominator is $2 \times 9.109 \times 2.998 \times 10^{-23} = 54.62 \times 10^{-23} = 5.462 \times 10^{-22} \text{ kg m/s}$, so

$$a_0 = \frac{1.0546 \times 10^{-34}}{5.462 \times 10^{-22}} \text{ m} = 1.931 \times 10^{-13} \text{ m}.$$

This is about 193 times the proton scale of 10^{-15} m and about 0.00365 times the Bohr radius $a_{\text{Bohr}} = 5.29 \times 10^{-11} \text{ m}$ (ratio $1.931/529 = 3.65 \times 10^{-3}$): the rest-frame trembling is a nuclear-scale modulation of an atomic-scale wave packet, invisible as a displacement but present in the wave-packet structure.

Step 5 — Illustrative gapped Dirac material. The beat-frequency statement (3.6) applies to any two-eigenspace Hamiltonian. Take an illustrative material with band edges at $\pm\Delta$, $\Delta = 0.1 \text{ eV}$, so $E_+ - E_- = 2\Delta = 0.2 \text{ eV}$. The frequency is

$$f = \frac{E_+ - E_-}{h} = \frac{2\Delta}{2\pi\hbar_{\text{eV s}}} = \frac{0.2 \text{ eV}}{6.2832 \times 6.582 \times 10^{-16} \text{ eV s}}.$$

The denominator is $6.2832 \times 6.582 \times 10^{-16} = 41.36 \times 10^{-16} = 4.136 \times 10^{-15} \text{ eV s}$, so

$$f = \frac{0.2}{4.136 \times 10^{-15}} \text{ Hz} = 4.836 \times 10^{13} \text{ Hz}.$$

This lies in the tens of terahertz, i.e. the range of ultrafast optical pump–probe techniques, which is precisely why simulator and material platforms can resolve the oscillation that the free electron cannot show directly. The corresponding period is $T = 1/f = 2.068 \times 10^{-14} \text{ s} \approx 20.7 \text{ fs}$.

Step 6 — Scaling law. Because (3.6) is linear in the gap, the frequency for any two-eigenspace system scales as

$$f = \frac{E_+ - E_-}{h} = (2.418 \times 10^{14} \text{ Hz/eV}) \times (E_+ - E_-),$$

where the conversion factor follows from $1/(4.136 \times 10^{-15} \text{ eV s}) = 2.418 \times 10^{14} \text{ Hz/eV}$ (check: $2.418 \times 10^{14} \times 0.2 = 4.836 \times 10^{13} \text{ Hz}$, matching Step 5). This single line converts any measured ZB frequency in any platform into a gap measurement, and conversely predicts the oscillation frequency from any independently known gap.

5. Results

The derivation of Section 3 and the arithmetic of Section 4 yield the following results, each traceable to shown steps.

R1 (Exact operator solution). From the single algebraic input $\{H, \alpha_j\} = 2cp_j$ (Eq. 3.1), the velocity operator evolves exactly as $\alpha_j(t) = cp_j H^{-1} + Z_j e^{-2iHt/\hbar}$ with $Z_j = \alpha_j - cp_j H^{-1}$ anticommute with H (Eq. 3.3), and the position decomposes into drift, static shift, and oscillation as in Eq. (3.4). This holds for every representation of the Clifford relations.

R2 (Beat-frequency theorem). For any Hamiltonian on a Clifford module whose velocity operators are Clifford generators satisfying (3.5), the observable oscillation exists if and only if two eigenspaces are coherently populated, and its angular frequency is $\omega_{\text{ZB}} = (E_+ - E_-)/\hbar$ (Eq. 3.6). A state prepared entirely within one eigenspace shows no oscillation.

R3 (Rest-frame electron numbers). With $c = 2.998 \times 10^8 \text{ m/s}$, $m_e = 9.109 \times 10^{-31} \text{ kg}$, $\hbar = 1.0546 \times 10^{-34} \text{ J s}$ (Steps 2–4): rest energy $m_e c^2 = 8.187 \times 10^{-14} \text{ J} = 0.5111 \text{ MeV}$; rest-frame frequency $f_0 = 2.471 \times 10^{20} \text{ Hz}$ (period $4.047 \times 10^{-21} \text{ s}$); amplitude scale $a_0 = \hbar/(2m_e c) = 1.931 \times 10^{-13} \text{ m}$.

R4 (Material-scale numbers). For the illustrative gapped Dirac material with band edges $\pm\Delta$, $\Delta = 0.1$ eV (Step 5): $f = 4.836 \times 10^{13}$ Hz, period ≈ 20.7 fs — resolvable by ultrafast probes.

R5 (Universal scaling). The frequency–gap conversion $f = 2.418 \times 10^{14}$ Hz/eV $\times (E_+ - E_-)$ (Step 6) is a direct consequence of the beat-frequency structure and is independent of dimension, representation, and platform.

R6 (Suppression prediction). Eigenspace filtering — preparing a wave packet wholly in the positive- or wholly in the negative-energy sector — eliminates the oscillatory term identically, in any platform realizing the structure (Section 3.4). This is the falsifiable signature carried into Sections 6 and 7.

6. Discussion

The hierarchy conjecture is a claim about generality, and its standing depends on how it handles its own failure modes.

What would falsify the main result. The central claim is that ZB-type oscillation is owned by the algebra: it occurs exactly when a velocity operator fails to commute with the Hamiltonian through Clifford structure and two eigenspaces are coherently populated. A single observation would falsify it: a platform exhibiting a sustained position oscillation at a frequency unrelated to any inter-eigenspace splitting — for example, an oscillation persisting after the state is prepared wholly within one eigenspace of the Hamiltonian, or an oscillation whose frequency does not equal $(E_+ - E_-)/\hbar$ for the eigenspaces actually populated. Conversely, the derivation of Section 3.4 shows that single-eigenspace states have identically vanishing cross term; a platform where filtering fails to suppress the oscillation would refute the claim that the oscillation is two-eigenspace interference. A second falsification path is representational: if some representation of the same Clifford relations produced different oscillation frequencies for the same physical state, the claim that (3.1) is representation-independent would fail; but (3.1) was derived from the relations alone, so this path is closed analytically and only the experimental filtering test remains decisive.

Limitations. The result rests on the following assumptions, with sensitivity noted for each:

- *Clifford anticommutator (3.1).* The entire operator solution follows from $\{H, \alpha_j\} = 2cp_j$; if a platform's velocity operators satisfy a different algebraic relation, the specific solution (3.3) does not apply, though the beat-frequency theorem (R2) covers any Hamiltonian with Clifford-generator velocities. The result is maximally sensitive to this input.
- *Invertibility of H .* The solution uses H^{-1} ; a gapless Hamiltonian (zero mode) requires a limiting treatment not carried out here.
- *Discrete, non-degenerate two-sector spectrum.* The clean beat frequency assumes exactly two relevant eigenspaces; additional degenerate or near-degenerate eigenspaces produce multi-frequency beating not analyzed here.
- *Free Hamiltonian.* Interactions, traps, and drives — present in the platforms of [2] and [4], where the Zeeman field and trap modify amplitudes and frequencies — are outside the free derivation; the theorem applies to them only insofar as their effective Hamiltonians retain the structure (3.5).
- *Constant inputs.* All numbers inherit the four-significant-figure precision of the stated constants; no measurement is claimed.
- *Curved backgrounds.* The research idea proposes extension to curved backgrounds; this paper does not perform it, and the generalization to position operators on curved space is open.
- *Non-Archimedean settings.* Entries [12–14] propose ultrametric state-space geometry under which the Archimedean beat-frequency statement would require re-derivation; this paper does not address it.

Arguing against the conjecture. The cautionary parallel of [9] applies with force: many Chern-Simons theories admit reformulations in which the metric 3-algebra never appears, showing that an "algebraic origin" can be a choice of description rather than a uniqueness claim. A skeptic can equally say that the two-eigenspace interference account of ZB is standard wave mechanics — any two coherently populated eigenspaces of any operator beat at their splitting — and that the Clifford structure is incidental scaffolding. The reply is that the Clifford anticommutator (3.1) is what forces the velocity operator out of the commutant of H in the first place, so the algebra determines *which* systems exhibit the effect; but the skeptic's reading is not excluded by anything derived here, and distinguishing "algebra-owned" from "generic two-level

beating" would require a platform with ZB-like oscillation lacking Clifford structure — which, if found, would itself narrow the conjecture to the weaker spectral claim. Open questions: the curved-background extension; the gapless case; multi-sector beating; and the ultrametric re-derivation flagged above.

7. Practical Implications and Engineering Pathways

(a) Who would use the result and what changes. The users are experimental teams running quantum simulators of Dirac-like dynamics: cold-atom groups implementing spin-orbit-coupled or tripod-scheme laser systems of the kind analyzed in [2] and [4], and groups engineering gapped Dirac-like band structures in materials or trapped ions. If the result holds, two decisions change. First, state preparation: an experimentalist who wants ZB populates both eigenspaces coherently; one who wants a clean single-sector wave packet filters one eigenspace out, and the oscillation is gone by construction rather than by tuning. Second, metrology: the oscillation frequency becomes a gap measurement via the conversion $f = 2.418 \times 10^{14} \text{ Hz/eV} \times (E_+ - E_-)$ (Step 6), usable as a cross-check on independently measured band gaps. Where no specific laboratory is named, the work is foundational: it supplies the criterion and the falsification test, not a device.

(b) Readiness. This is purely analytical work, not peer reviewed: under the NASA Technology Readiness Level definitions it sits at TRL 1–2 (basic principles observed and formulated; concept and application articulated analytically). No experimental demonstration was performed here, and analytical work does not exceed TRL 3; no claim of device or manufacturing readiness is made.

(c) Smallest next test. The minimal de-risking experiment uses an existing simulator platform realizing a two-band Dirac-like Hamiltonian: (i) prepare a wave packet as a coherent superposition of the two bands and measure the oscillation frequency, checking it against $f = 2.418 \times 10^{14} \text{ Hz/eV} \times (E_+ - E_-)$ using the independently known band gap; (ii) re-prepare the same wave packet filtered into a single band and verify the oscillation vanishes. Outcome (i) failing the frequency relation, or (ii) showing a residual oscillation of comparable amplitude, refutes the conjecture on that platform. The platforms of [2] and [4], which already demonstrate control over ZB amplitude and frequency through external fields and mirror oscillation, are the natural settings, since they already possess the state-manipulation tools the test requires.

(d) Binding constraints. In order of likely first contact: state-preparation fidelity — the suppression test requires eigenspace filtering with leakage well below the oscillation amplitude, and any unfiltered residual component produces a residual oscillation that could be misread as a falsification; decoherence and damping — the platforms of [2] and [4] report damping of the oscillation, which limits the observation window and must be separated from genuine suppression; frequency resolution — the measured oscillation frequency must be resolved against the drive and trap frequencies that [2] shows can shift the observed oscillation frequencies; and for material platforms, disorder broadening of the band edges, which blurs the gap entering the frequency prediction. Cost drivers and scale-up are not derivable from the material in this paper and are left as open questions.

(e) Order-of-magnitude performance bounds (projections). Labeled projections, with assumptions stated: assuming a platform realizes a two-eigenspace Hamiltonian with gap $E_+ - E_-$ known to within 1%, the beat-frequency prediction (Step 6) predicts the oscillation frequency to within 1%; for gaps in the range 0.01–1 eV this gives frequencies 2.4×10^{12} – $2.4 \times 10^{14} \text{ Hz}$, i.e. sub-picosecond ($\approx 0.42 \text{ ps}$) to ≈ 4.2 -femtosecond periods, within reach of ultrafast probe repetition and stroboscopic simulation protocols. These bounds follow only from the conversion of Step 6 and the assumed gap range; no platform-specific performance is claimed.

(f) Risks and misuse paths. No dual-use risk is apparent: the result concerns the interpretation of simulator dynamics. The main misuse is interpretive rather than physical — reading any oscillation in a driven simulator as "Zitterbewegung" without checking the two-eigenspace structure, which the criterion here is designed to prevent.

8. Conclusion

The Zitterbewegung oscillation is owned by the algebra. From the single Clifford anticommutator $\{H, \alpha_j\} = 2cp_j$ — derived from the relations, not from any representation — the exact operator solution $\alpha_j(t) = cp_j H^{-1} + Z_j e^{-2iHt/\hbar}$ follows, the position splits into drift, shift, and interference, and the observable oscillation occurs exactly when two eigenspaces are coherently populated, at the beat frequency $\omega_{\text{ZB}} = (E_+ - E_-)/\hbar$. Calculus produces the solution, trigonometry notates it, and geometry only pictures it. The numerical consequences are concrete: $f_0 = 2.471 \times 10^{20}$ Hz and amplitude 1.931×10^{-13} m for the rest-frame electron, and $f = 4.836 \times 10^{13}$ Hz for an illustrative 0.1-eV-gap Dirac material, with the universal conversion $f = 2.418 \times 10^{14} \text{ Hz/eV} \times (E_+ - E_-)$. The claim converts ZB from a Dirac-equation curiosity into a spectral diagnostic and a design rule — populate two eigenspaces to create it, filter to remove it — and it stands or falls on a single test: eigenspace-filtered wave-packet dynamics on an existing simulator platform.

Provenance and AI Involvement

- This paper was drafted, reconciled, reviewed, revised and checked by the QNFO automated research pipeline (qnfo-research-exec 0.16.0-revise-retry). Language models were tools in that process; they are not authors. The named author is accountable for the content.
- Models: drafting by openai/gpt-oss-120b, zai-org/glm-5.3-flash; reconciliation, completion, review, revision and verification by zai-org/glm-5.3-flash, zai-org/glm-5.3, openai/gpt-oss-120b, with the QNFO model router as a fallback.
- Sources: every reference is drawn from a bibliography retrieved from arXiv and the QNFO corpus before drafting; statements about a cited work are restricted to its retrieved title and abstract.
- Checks passed before publication: required sections present and finished, cross-references resolve, reference list rendered from the retrieved bibliography, no assistant residue or placeholders, adversarial model review with its blocking findings resolved, and a citation existence check at publication.
- Independent recomputation: a separately generated script recomputed 11 quantitative claims from the paper's stated inputs; 0 differed from the stated value at the stated precision.
- Not checked: whether each cited source supports the sentence that cites it beyond its abstract; derivation steps other than the recomputed claims. This paper has not been peer reviewed, and this version was published by the pipeline without a prior manual review by the author.

References

- [1] Converting Zitterbewegung Oscillation to Directed Motion. arXiv:1105.2884. <https://arxiv.org/abs/1105.2884> [2] Zitterbewegung effect in spin-orbit coupled spin-1 ultracold atoms. arXiv:1210.5030. <https://arxiv.org/abs/1210.5030> [3] Quantum Probability's Algebraic Origin. arXiv:2009.08489. <https://arxiv.org/abs/2009.08489> [4] Driven Dirac-like Equation via Mirror Oscillation: Controlled Cold-Atom Zitterbewegung. arXiv:1003.3074. <https://arxiv.org/abs/1003.3074> [5] An algebraic origin for quark confinement. arXiv:hep-th/0307046. <https://arxiv.org/abs/hep-th/0307046> [6] Does the Coulomb potential have an algebraic origin?. arXiv:2105.02115. <https://arxiv.org/abs/2105.02115> [7] Affine Lie Algebraic Origin of Constrained KP Hierarchies. arXiv:hep-th/9408104. <https://arxiv.org/abs/hep-th/9408104> [8] On the Hopf algebraic origin of Wick normal-ordering. arXiv:hep-th/0007032. <https://arxiv.org/abs/hep-th/0007032> [9] On the Lie-algebraic origin of metric 3-algebras. arXiv:0809.1086. <https://arxiv.org/abs/0809.1086> [10] Some integrable systems of algebraic origin and separation of variables. arXiv:1712.04422. <https://arxiv.org/abs/1712.04422> [11] New simple groups with a BN-pair. arXiv:1306.0174. <https://arxiv.org/abs/1306.0174> [12] QNFO: Zitterbewegung: From Archimedean Puzzle to Adelic Observable [13] QNFO: Five Pillars, One Structure: Consilient Convergence in QNFO Research [14] QNFO: The Ultrametric Foundation: A Unified Thesis on Number, Time, Knowledge, and Computation