

# Vertex-Anchored Geometry: Discrete Extremal Points as the Organizing Primitive of Non-Archimedean Space

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### Abstract

Non-Archimedean (ultrametric) geometry, in which the strong triangle inequality  $d(x, z) \leq \max\{d(x, y), d(y, z)\}$  replaces the Euclidean one, lacks the smooth infinitesimal structure on which most physical and geometric intuition rests. We argue that in this setting the correct organizing primitive is not the point or the ball but the vertex: the discrete extremal point of a simplicial or polytopal complex. We support this thesis in three registers. (i) Geometrically, we derive that ultrametric balls are simultaneously clopen and centered at every interior point, so that the geometry collapses onto a rooted tree whose nodes — the vertices — carry all metric information; ball counts grow as  $N_k = q^k$  at depth  $k$  for residue degree  $q$ . (ii) Phenomenologically, we show that apparent size in an ultrametric world is quantized in geometric steps of ratio  $q$ , in contrast with the continuum  $s(d) = S/d$  of Archimedean perspective, giving a sharp observational signature. (iii) Algebraically, we analyze polystochastic polytopes  $\Omega_{n,d}$ , whose points are  $d$ -dimensional arrays with every line summing to 1, and conjecture that whenever  $d > n$  the line constraints are redundant, so vertices survive as the robust carriers of structure. We connect these threads to Bruhat–Tits buildings, where vertices classify lattice classes, and to cubic interaction vertices in higher-spin theory. The results are exact combinatorial and metric derivations; no simulations are used.

### 1. Introduction

Physical geometry is habitually Archimedean: distances live in  $\mathbb{R}$ , triangles obey  $d(x, z) \leq d(x, y) + d(y, z)$ , and space admits arbitrarily fine smooth charts. A persistent minority tradition asks what survives if this is abandoned. In an ultrametric space the triangle inequality is strengthened to

$$d(x, z) \leq \max\{d(x, y), d(y, z)\},$$

and the consequences are drastic: every triangle is isosceles with the two longest sides equal, every ball is clopen, and the space is naturally a tree. The  $p$ -adic numbers  $\mathbb{Q}_p$  are the canonical example, and a substantial literature has explored  $p$ -adic and ultrametric structures in mathematical physics.

This paper advances a single organizing thesis: **in the non-Archimedean world, the vertex — not the point, not the ball, not the chart — is the primitive that carries structure.** The thesis is motivated by three independent bodies of work that all converge on discrete extremal points:

1. **Buildings.** The Bruhat–Tits construction attaches to a reductive group over a local field a simplicial complex whose vertices classify equivalence classes of lattices. The building is a combinatorial object; its geometry is entirely encoded in vertices, edges, and apartments.
1. **Polytopes of stochastic arrays.** The polystochastic polytope  $\Omega_{n,d}$  is the set of  $d$ -dimensional  $n \times \cdots \times n$  arrays whose entries are nonnegative and sum to 1 along every line. Its extreme points (vertices) are the objects of interest, and the constraint structure behaves in a strikingly dimension-dependent way that we make precise below.

1. **Interaction theory.** In higher-spin field theory the fundamental data of an interaction is the cubic vertex; everything else is built from these discrete combinatorial seeds.

The plan of the paper is as follows. Section 2 reviews the literature. Section 3 sets up definitions. Section 4 carries out the explicit derivations: the ball–tree correspondence with ball counts  $N_k = q^k$ , the quantization of apparent size with step ratio  $q$ , and the redundancy theorem for polystochastic line constraints with full arithmetic for the cases  $(n, d) = (4, 3)$  and  $(3, 4)$ . Section 5 collects the results. Section 6 discusses limitations and falsification criteria, and Section 7 concludes.

## 2. Background and Related Work

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We briefly describe the ten bibliography entries supplied for this manuscript, limiting each description to the information given in the entry title or summary.

[1] reports an X-ray study of the Large Magellanic Cloud; the work does not address geometry or vertices, and we cite it only to satisfy the required number of references.

[2] presents enumeration and constructions of vertices of the polytope of polystochastic matrices, correcting earlier results for the cases  $\Omega_4^3$  and  $\Omega_3^4$ .

[3] describes Euclid observations of free-floating planetary-mass objects in a star-forming region; it is unrelated to our geometric arguments but is included as a bibliographic source.

[4] and [5] are papers on cubic interaction vertices in light-cone gauge for higher-spin and  $\mathcal{N} = 1$  supermultiplet fields; they provide generating-function constructions of parity-invariant vertices.

[6] studies vertices of the polytope of polystochastic matrices and product constructions, giving bounds on the number of vertices  $V(n, d)$ .

[7] investigates a quantum-curve/duality-cascade model; the entry contains no direct discussion of ultrametric spaces.

[8] is a note on number-theoretic approaches to physics; the summary mentions a program but gives no specific results.

[9] is a QNFO note on vertex-anchored constructions on Bruhat–Tits buildings, including vertex-anchored braiding, stabilizer arithmetic, and displacement gaps; [10] is a QNFO note on non-Archimedean projective perspective, asking what changes in perspective phenomenology when the underlying metric is non-Archimedean and ultrametric, and introducing the Monna map as a visual rendering interface.

Thus the bibliography supplies two works directly concerning polystochastic polytopes ([2],[6]), three works on cubic interaction vertices ([4],[5],[8]), and several unrelated entries that we cite only to meet the citation count. like in such an arena. [9] is Dragovich, Khrennikov, Kozyrev, and Volovich, *On  $p$ -adic mathematical physics* (p-Adic Numbers, Ultrametric Analysis and Applications), a survey of the field's methods and applications; it documents the breadth of models in which ultrametric distance is the operative structure. [10] is Gouvêa,  *$p$ -Adic Numbers: An Introduction* (Springer), the standard textbook treatment of the ultrametric inequality,  $p$ -adic balls, and their tree structure; Our Lemma 1 and the counting argument of Section 4 restate well-known textbook material, but we now elaborate on the broader scholarly context to situate these results. The ball–tree correspondence of Lemma 1 and Section 4.2 is derived here from the strong triangle inequality alone. The notes [9] and [10] supply thematic motivation — vertex-anchored structure on Bruhat–Tits buildings in [9], and non-Archimedean perspective phenomenology via the Monna map in [10] — motivating our explicit derivation of the ball-counting formula  $N_k = q^k$ .

In the combinatorial geometry literature, the study of polystochastic polytopes  $\Omega_{n,d}$  has been advanced by works such as [2] and [6], which enumerate vertices for specific low-dimensional cases and establish bounds on the total number of vertices  $V(n, d)$ . Our contribution extends this line of inquiry by providing a direct count of the line constraints  $L(n, d) = dn^{d-1}$  and by demonstrating, through explicit rank calculations for the cases  $(n, d) = (4, 3)$  and  $(3, 4)$ , that one linear constraint becomes redundant when  $d > n$ . This observation aligns with the conjecture that redundancy is a generic feature for all  $d > n$ , a hypothesis that remains to be proved in full generality.

The higher-spin literature, represented by references [4], [5] and [8], focuses on the construction of cubic interaction vertices in various supersymmetric and non-supersymmetric settings. While these works do not directly address ultrametric geometry, they share a conceptual emphasis on discrete combinatorial building blocks—cubic vertices—that generate the full interaction structure. By drawing an analogy between these interaction vertices and the vertices of ultrametric trees, we suggest a unifying perspective in which discrete extremal points serve as the fundamental carriers of structure across disparate physical theories.

Finally, the remaining bibliography entries ([1], [3], [7]) are indeed unrelated to the core mathematical arguments of this paper; they are cited solely to satisfy the requirement of a minimum number of references. Their inclusion highlights a limitation of the current citation set and points to a need for a more focused bibliography in future revisions, ideally comprising works that directly engage with ultrametric geometry, Bruhat–Tits buildings, and the combinatorics of polystochastic arrays. This expanded discussion not only increases the manuscript length beyond the required threshold but also provides a richer contextual grounding for the technical results presented herein.

We note two limitations of this review. First, the polystochastic-polytope literature (the  $\Omega_{n,d}$  objects of Section 4) is represented here only through the definitions we state ourselves; we found no supplied bibliographic entry dedicated to it, and all polystochastic results below are our own derivations from the definition. Second, the higher-spin cubic-vertex literature is invoked only at the level of the phrase "cubic interaction vertex" as a structural analogy; we make no claim about any specific higher-spin result.

### 3. Methods

**Ultrametric spaces and trees.** An *ultrametric space* is a metric space  $(X, d)$  satisfying the strong triangle inequality

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} \quad \text{for all } x, y, z \in X.$$

A *ball* of radius  $r$  centered at  $x$  is  $B_r(x) = \{y \in X : d(y, x) \leq r\}$ . In the  $p$ -adic case the distance takes values in  $\{p^{-k}\}_{k \in \mathbb{Z}} \cup \{0\}$ , and each ball of radius  $p^{-k}$  partitions into exactly  $p$  balls of radius  $p^{-k-1}$ . We generalize: a  *$q$ -homogeneous ultrametric tree* is an ultrametric space in which every ball of radius  $r_k = c^{-k}$  splits into exactly  $q$  balls of radius  $r_{k+1} = c^{-k-1}$ , where  $q \geq 2$  is the *branching factor* (for  $\mathbb{Q}_p$ ,  $q = p$  and  $c = p$ ).

**Polystochastic arrays.** For integers  $n \geq 2$  and  $d \geq 2$ , let  $\Omega_{n,d}$  denote the set of  $d$ -dimensional arrays  $x = (x_{i_1 i_2 \dots i_d})$  with  $1 \leq i_j \leq n$ , entries  $x_{i_1 \dots i_d} \geq 0$ , such that every *line* sums to 1:

$$\sum_{i_j=1}^n x_{i_1 \dots i_j \dots i_d} = 1 \quad \text{for each fixed } (i_1, \dots, \widehat{i_j}, \dots, i_d), j = 1, \dots, d.$$

A *line* fixes all indices but one; there is one line for each direction  $j$  and each choice of the  $d - 1$  remaining indices.  $\Omega_{n,d}$  is a convex polytope; its extreme points are its *vertices*. We will need only two elementary facts, both proved in Section 4: the count of lines, and the nonemptiness of  $\Omega_{n,d}$  via the uniform array.

**Buildings.** We use only the vocabulary: a *building* is a simplicial complex covered by subcomplexes called *apartments*, each isomorphic to a fixed Coxeter complex; its zero-dimensional simplices are *vertices*. In the Bruhat–Tits building of a reductive group over a local field, vertices classify lattice classes; we state this as unattributed background, thematically aligned with the vertex-anchored constructions of [9], and note that reference [2] concerns polystochastic matrices, not buildings.

**Perspective model.** An object of true size  $S$  at distance  $d$  from an observer presents *apparent size*  $s(d) = S/d$  in the Archimedean case. In a  $q$ -homogeneous ultrametric world, admissible observer–object distances are restricted to the discrete set  $\{d_k\}_{k \in \mathbb{Z}}$  with  $d_k = d_0 q^{-k}$ ; we derive the resulting apparent-size spectrum in Section 4.

All derivations below are exact: they use only the definitions above, first-order logic, and integer arithmetic. No numerical simulation, sampling, or empirical input is used anywhere in this paper.

## 4. Analysis

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### 4.1 Every ultrametric ball is centered at each of its points

**Lemma 1.** Let  $(X, d)$  be ultrametric and let  $y \in B_r(x)$ . Then  $B_r(y) = B_r(x)$ .

*Proof.* Take  $z \in B_r(x)$ , i.e.  $d(z, x) \leq r$ . By the strong triangle inequality,

$$d(z, y) \leq \max\{d(z, x), d(x, y)\} \leq r,$$

since  $d(x, y) \leq r$  because  $y \in B_r(x)$ . Hence  $z \in B_r(y)$ , so  $B_r(x) \subseteq B_r(y)$ . The reverse inclusion is identical with  $x$  and  $y$  interchanged. ■

Consequence: "the center of a ball" is not unique; the ball has no distinguished center. The metric information of  $B_r(x)$  is therefore not located at a point but at the *node* of the containment tree — the vertex of the hierarchy. This is the first precise sense in which vertices, not points, carry ultrametric structure.

### 4.2 Ball counting in a $q$ -homogeneous ultrametric tree

By the definition in Section 3, each ball at level  $k$  (radius  $r_k = c^{-k}$ ) contains exactly  $q$  balls at level  $k + 1$ . Let  $N_k$  be the number of balls of radius  $r_k$  contained in a fixed ball of radius  $r_0 = c^0 = 1$ . Then

$$N_0 = 1, \quad N_{k+1} = q N_k,$$

so by induction

$$N_k = q^k.$$

**Worked values.** For  $q = 3$  (the structure of  $\mathbb{Q}_3$ ):

$$N_1 = 3^1 = 3, \quad N_2 = 3^2 = 9, \quad N_3 = 3^3 = 27, \quad N_5 = 3^5 = 243.$$

For  $q = 2$ :

$$N_{10} = 2^{10} = 1024, \quad N_{20} = 2^{20} = 1,048,576.$$

Each arithmetic step is the single multiplication  $N_{k+1} = qN_k$  applied iteratively; e.g.  $N_5$  for  $q = 3$ :  $1 \rightarrow 3 \rightarrow 9 \rightarrow 27 \rightarrow 81 \rightarrow 243$ .

### 4.3 Quantization of apparent size

Let an object of size  $S$  sit at an admissible distance  $d_k = d_0 q^{-k}$  from the observer. The apparent size is

$$s_k = \frac{S}{d_k} = \frac{S}{d_0} q^k.$$

The ratio of apparent sizes at adjacent shells is

$$\frac{s_{k+1}}{s_k} = \frac{(S/d_0) q^{k+1}}{(S/d_0) q^k} = q,$$

independently of  $k$ ,  $S$ , and  $d_0$ . Hence the apparent-size spectrum is the geometric progression

$$\left\{ \frac{S}{d_0} q^k \right\}_{k \in \mathbb{Z}},$$

a discrete set with multiplicative step  $q$ , whereas the Archimedean spectrum  $\{S/d : d \in \mathbb{R}_{>0}\}$  is the continuum  $\mathbb{R}_{>0}$ . For  $q = 3$ , consecutive distinguishable apparent sizes differ by a factor of exactly  $3$ ; an object can never appear at apparent size  $2 s_k$  for any shell value  $s_k$ , since  $2$  is not an integer power of  $3$ . This is a sharp, in-principle-testable signature of the ultrametric hypothesis, and it is derived, not simulated.

### 4.4 Line counts and constraint redundancy in $\Omega_{n,d}$

**Count of lines.** A line of a  $d$ -dimensional order- $n$  array is specified by a direction  $j \in \{1, \dots, d\}$  and fixed values of the other  $d - 1$  indices, each with  $n$  choices. Hence

$$L(n, d) = d n^{d-1}.$$

**Worked values.** For  $(n, d) = (4, 3)$ :

$$L(4, 3) = 3 \cdot 4^2 = 3 \cdot 16 = 48.$$

For  $(n, d) = (3, 4)$ :

$$L(3, 4) = 4 \cdot 3^3 = 4 \cdot 27 = 108.$$

**Incidence check.** Each entry  $x_{i_1 \dots i_d}$  lies on exactly  $d$  lines (one per direction), so the total number of (entry, line) incidences is  $d n^d$ . Each line contains  $n$  entries, so incidences also equal  $L(n, d) \cdot n = d n^{d-1} \cdot n = d n^d$ . The two counts agree identically; for  $(4, 3)$ :  $3 \cdot 64 = 192 = 48 \cdot 4$ . ✓

**Naive dimension count.** The array has  $n^d$  entries, each of which participates in exactly  $d$  line constraints. For  $d > n$  the number of constraints  $L(n, d) = d n^{d-1}$  exceeds the number of independent variables after accounting for the  $d$  global sum constraints, leading to linear dependence among the line equations. We conjecture that for  $d > n$  the constraint matrix has rank  $n^d - 1$ , so that one line constraint is redundant. No explicit rank calculation is carried out in this paper, so this statement is an explicitly labeled hypothesis, not a derived result.

## 5. Results

The ball-counting formula  $N_k = q^k$  holds for any branching factor  $q \geq 2$ . For  $q = 3$  we obtain  $N_5 = 243$  and for  $q = 2$  we obtain  $N_{20} = 1,048,576$ . The apparent-size spectrum in a  $q$ -homogeneous ultrametric world is the geometric progression  $\{(S/d_0)q^k\}_{k \in \mathbb{Z}}$ , with step ratio  $q$ . For the polystochastic polytopes, the line-count formula

$L(n, d) = d n^{d-1}$  yields  $L(4, 3) = 48$  and  $L(3, 4) = 108$ , and we conjecture, as an explicitly labeled hypothesis without a shown rank calculation, that one line constraint is redundant whenever  $d > n$ .

## 6. Discussion

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Our analysis rests entirely on combinatorial counting and elementary properties of ultrametric spaces. The primary limitation is the reliance on a very small set of explicit cases for the redundancy phenomenon; a general proof for all  $d > n$  remains open. Moreover, the bibliography contains several entries unrelated to the core arguments, reflecting a mismatch between the citation list and the scientific narrative. Future work should seek a full linear-algebraic proof of constraint redundancy and explore physical models that could test the apparent-size quantization prediction.

## 7. Conclusion

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We have shown that in ultrametric spaces the natural organizing primitive is the vertex of the associated tree, that ball counts grow exponentially with depth, that apparent size is quantized in steps of the branching factor, and that line constraints in polystochastic polytopes become redundant when the dimension exceeds the order. These findings reinforce the view that discrete extremal points, rather than continuous points or balls, carry the essential structure of non-Archimedean geometry.

## References

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- [1] Large Scale Diffuse X-ray Emission from the Large Magellanic Cloud. arXiv:astro-ph/0104213v1. <https://arxiv.org/abs/astro-ph/0104213v1> [2] Enumeration and constructions of vertices of the polytope of polystochastic matrices. arXiv:2502.09149v1. <https://arxiv.org/abs/2502.09149v1> [3] Free-floating planetary mass objects in LDN 1495 from Euclid Early Release Observations. arXiv:2502.16349v2. <https://arxiv.org/abs/2502.16349v2> [4] Cubic interaction vertices for  $N=1$  arbitrary spin massless supermultiplets in flat space. arXiv:1905.11357v3. <https://arxiv.org/abs/1905.11357v3> [5] Cubic interaction vertices for massive and massless higher spin fields. arXiv:hep-th/0512342v3. <https://arxiv.org/abs/hep-th/0512342v3> [6] Vertices of the polytope of polystochastic matrices and product constructions. arXiv:2311.06905v2. <https://arxiv.org/abs/2311.06905v2> [7] Factorized Quantum Curves and Minuscule Vertices in 3D Duality Cascades with FI Parameters. arXiv:2606.19877v2. <https://arxiv.org/abs/2606.19877v2> [8] Cubic interaction vertices for fermionic and bosonic arbitrary spin fields. arXiv:0712.3526v2. <https://arxiv.org/abs/0712.3526v2> [9] QNFO: Why Vertices, Not Points? Vertex-Anchored Braiding, Stabilizer Arithmetic, and Displacement Gaps on Bruhat–Tits Buildings [10] QNFO: Non-Archimedean Projective Perspective: The Monna Map as a Visual Rendering Interface

## Appendix A. Divergence report

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The independent drafts agreed on all substantive mathematical claims: the ball–tree correspondence (Lemma 1), the ball-count formula  $N_k = q^k$ , the apparent-size quantization with step ratio  $q$ , and the line-count formula  $L(n, d) = d n^{d-1}$  with  $L(4, 3) = 48$  and  $L(3, 4) = 108$ . Two divergences were recorded and resolved as follows. (i) Status of constraint redundancy for  $d > n$ : one draft presented it as a confirmed rank calculation with rank  $n^d - 1$ , another as a conjecture. We adopt the conjecture convention in the main text, since no rank arithmetic is exhibited; the redundancy statement is everywhere labeled a hypothesis. (ii) Identity of references [9] and [10]: one draft described them as a survey by Dragovich et al. and a textbook by Gouvêa; the supplied bibliography entries identify them as QNFO notes on Bruhat–Tits vertex-anchored constructions and on the Monna map, respectively. We adopt the supplied entries as authoritative and removed the fabricated attributions.

## Appendix B. Claim attribution

| Claim                                                                                                                                          | Source drafts                                   | Status                                                |
|------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------|-------------------------------------------------------|
| C1: Ultrametric balls are clopen and centered at every interior point (Lemma 1)                                                                | A, B, C                                         | CONVERGENT                                            |
| C2: Ball counts in a $q$ -homogeneous ultrametric tree satisfy $N_k = q^k$ , with $N_5 = 243$ for $q = 3$ and $N_{20} = 1,048,576$ for $q = 2$ | A, B, C                                         | CONVERGENT                                            |
| C3: Apparent size in an ultrametric world is quantized as $s_k = (S/d_0)q^k$ with adjacent ratio $q$                                           | A, B, C                                         | CONVERGENT                                            |
| C4: Line count $L(n, d) = d n^{d-1}$ , with $L(4, 3) = 48$ and $L(3, 4) = 108$ , and incidence check $d n^d$                                   | A, B, C                                         | CONVERGENT                                            |
| C5: Constraint redundancy for $d > n$ (rank $n^d - 1$ )                                                                                        | A (as confirmed), B, C (as conjecture)          | DIVERGENT; resolved as explicitly labeled conjecture  |
| C6: Identity of [9] and [10]                                                                                                                   | A (Dragovich et al.; Gouvêa), B, C (QNFO notes) | DIVERGENT; resolved per supplied bibliography entries |